Engineering Mathematics – II STV Gems publications

APRIL 2016

Time – Three hours

(Maximum Marks: 75)

[N.B:- (1) Answer any FIVE questions in each of PART-A& PART-B and any two divisions of each questions in PART-C

(2) Each questions carries 2(two)marks in PART-A ,3(three)marks in PART-B and 5(five)marks for each division in PART-C.]

PART – A

1. Find the centre and radius of the circle $x^2 + y^2 + 2x + 2y - 7 = 0$
2. Write down the condition for two circles $x^2 + y^2 + 2g_1x + 2f_1y + c_1 = 0$ and $x^2 + y^2 + 2g_2x + 2f_2y + c_2 = 0$ to cut orthogonally
3. Find the unit vector along the vector direction $2\vec{i} - \vec{j} - \vec{k}$
4. What are the values of (i) $\vec{i} \bullet \vec{i}$ (ii) $\vec{i} \times \vec{j}$
5. Evaluate: $[\vec{i}, \vec{j}, \vec{k}]$
6. Evaluate: $\int (x^2 + sec^2 x)\, dx$
7. Evaluate: $\int \frac{dx}{\sqrt{4-x^2}}$
8. Evaluate $\int_1^2 x^2\, dx$

PART – B

9. Find the equation of the circle passing through the point (1,1) and concentric with the circle $x^2 + y^2 + 4x + 6y - 15 = 0$

10. Find the equation of the parabola with its focus at (–1, –2) and $x + 2y = 0$ as its directrix

11. Find the modulus and direction cosines of the vector $2\vec{i} + 3\vec{j} + 4\vec{k}$

12. Find the value of 'm' if the vectors $2\vec{i} + m\vec{j} - 3\vec{k}$ and $3\vec{i} + \vec{j} + 4\vec{k}$ are perpendicular.

13. Prove that the vectors $2\vec{i} - \vec{j} + 2\vec{k}$, $5\vec{i} + 2\vec{j} + 3\vec{k}$ *and* $4\vec{i} - 2\vec{j} + 4\vec{k}$ are coplanar.

14. Evaluate: $\int cos^3 x\, dx$

15. Evaluate: $\int \frac{dx}{\sqrt{25-x^2}}$

16. Evaluate: $\int x\, e^{6x} dx$

PART- C

17. (a) Find the equation of the circle passing through the point A (1 , 2) and having its centre at C (4 , 6)

(b) Prove that the circles $x^2 + y^2 - 8x + 6y - 23 = 0$ and $x^2 + y^2 - 2x - 5y + 16 = 0$ cut orthogonally

(c) Find 'k' such that the equation $2x^2 + 3xy - 2y^2 - 5x + 5y + k = 0$ represents a pair of straight line

18. (a) Show that the points whose position vectors are $2\vec{i} + 4\vec{j} + 3\vec{k}$, $4\vec{i} + \vec{j} - 4\vec{k}$ and $6\vec{i} + 5\vec{j} - \vec{k}$ form a right angled triangle

(b) Prove that the vectors $\vec{i} + 2\vec{j} + \vec{k}, \vec{i} + \vec{j} - 3\vec{k}$ and $7\vec{i} - 4\vec{j} + \vec{k}$ are mutually perpendicular

(c) A particle acted on by forces $3\vec{i} + 2\vec{j} - 3\vec{k}$ and $\vec{i} + 7\vec{j} + 7\vec{k}$ is displaced from the point $\vec{i} + 2\vec{j} + 3\vec{k}$ to the point $3\vec{i} - 5\vec{j} + 4\vec{k}$. Find the total work done by the forces

19 (a) Find the area of triangle whose vertices are having position vectors $3\vec{i} + 2\vec{j} - \vec{k}, 2\vec{i} - 3\vec{j} + \vec{k}$ *and* $5\vec{i} + \vec{j} + 3\vec{k}$

(b) Find the moment about the point (4 , 3 , 2) of the force represented by $5\vec{i} - 4\vec{j} + 2\vec{k}$ acting through the point (2 , 1 , − 3)

(c) If $\vec{a} = 2\vec{i} - \vec{j} + 2\vec{k}, \vec{b} = \vec{i} + \vec{j} + \vec{k}, \vec{c} = \vec{i} + 2\vec{j} + 3\vec{k}$ and $\vec{d} = \vec{i} - \vec{j} - \vec{k}$ find$(\vec{a} \times \vec{b}) \bullet (\vec{c} \times \vec{d})$

20.(a) Evaluate: $(i) \int (x + 1)(2x - 3)\, dx$

$(ii) \int cos5x\ cos2x\ dx$

(b) Evaluate: $(i) \int (x^2 + x + 1)^5 (2x + 1)\, dx$

$(ii) \int \frac{\sec^2 x}{1+tanx}\, dx$

(c) Evaluate: $(i) \int \frac{dx}{x^2-36}$ $(ii) \int \frac{dx}{\sqrt{49-(x+5)^2}}$

21. (a) Evaluate: $(i) \int x \cos 3x\, dx$ $(ii) \int x^2 \log x\, dx$

(b) Evaluate: $(i) \int x^2 \sin 3x\, dx$ $(ii) \int x^2 e^{5x}\, dx$

(c) Evaluate: $\int_0^{\frac{\pi}{2}} (2 + \sin x)^3 \cos x\ dx$

ANSWERS

PART - A

1. Find the centre and radius of the circle

$x^2 + y^2 + 2x + 2y - 7 = 0$

Answer Given $x^2 + y^2 + 2x + 2y - 7 = 0$ ———(1)

We know that, $x^2 + y^2 + 2gx + 2fy + c = 0$ ———(2)

On comparing (1) & (2),we get

$2g = 2$	$2f = 2$	
$g = \frac{2}{2} = 1$	$f = \frac{2}{2} = 1$	$c = -7$

$\therefore$ Centre C = $(-g, -f) = (-1, -1)$

Radius r = $\sqrt{g^2 + f^2 - c}$

$$r = \sqrt{(1)^2 + (1)^2 - (-7)}$$

$$= \sqrt{1+1+7} = \sqrt{9} = 3 \text{ units}$$

2. Write down the condition for two circles

$x^2 + y^2 + 2g_1x + 2f_1y + c_1 = 0$ and $x^2 + y^2 + 2g_2x + 2f_2y + c_2 = 0$

to cut orthogonally

Answer The condition for two circles to cut orthogonally is

$$\mathbf{2g_1 g_2 + 2f_1 f_2 = c_1 + c_2}$$

3. Find the unit vector along the vector direction $2\vec{i}-\vec{j}-\vec{k}$

Answer

Let $\vec{a} = 2\vec{i}-\vec{j}-\vec{k} \Rightarrow x\vec{i}+y\vec{j}+z\vec{k}$

$$|\vec{a}| = \sqrt{x^2+y^2+z^2}$$

$$= \sqrt{2^2+(-1)^2+(-1)^2}$$

$$= \sqrt{4+1+1}$$

$$= \sqrt{6}$$

$$\therefore \text{ unit vector } \hat{a} = \frac{\vec{a}}{|\vec{a}|} = \frac{2\vec{i}-\vec{j}-\vec{k}}{\sqrt{6}}$$

4. What are the values of (i) $\vec{i}\bullet\vec{i}$ (ii) $\vec{i}\times\vec{j}$

Answer

(i) $\vec{i}\bullet\vec{i} = 1$ (ii) $\vec{i}\times\vec{j} = \vec{k}$

5. Evaluate: $[\vec{i},\vec{j},\vec{k}]$

Answer

$$[\vec{i},\vec{j},\vec{k}] = \begin{vmatrix}1 & 0 & 0\\0 & 1 & 0\\0 & 0 & 1\end{vmatrix}$$

$$= 1\begin{vmatrix}1 & 0\\0 & 1\end{vmatrix} - 0\begin{vmatrix}0 & 0\\0 & 1\end{vmatrix} + 0\begin{vmatrix}0 & 1\\0 & 0\end{vmatrix}$$

$$= 1(1-0) - 0(0-0) + 0(0-0)$$

$$[\vec{i},\vec{j},\vec{k}] = 1$$

6. Evaluate: $\int(x^2+sec^2x)\,dx$

Answer

$$Given \int(x^2+sec^2x)dx = \frac{x^3}{3} + tanx + c$$

7. Evaluate: $\int\frac{dx}{\sqrt{4-x^2}}$

Answer

$$\int\frac{dx}{\sqrt{4-x^2}} = \int\frac{dx}{\sqrt{2^2-x^2}} = sin^{-1}\left(\frac{x}{2}\right) + c$$

8. Evaluate: $\int_1^2 x^2\,dx$

Answer

$$\int_1^2 x^2\,dx = \left[\frac{x^3}{3}\right]_1^2$$

$$= \frac{2^3-1^3}{3}$$

$$= \frac{8-1}{3} = \mathbf{\frac{7}{3}}$$

PART - B

9. Find the equation of the circle passing through the point (1,1) and concentric with the circle $x^2 + y^2 + 4x + 6y - 15 = 0$

Answer Given $x^2 + y^2 + 4x + 6y - 15 = 0$

Concentric circle is $x^2 + y^2 + 4x + 6y + k = 0$ —— (1)

(1) passes through (1, 1) ⇒ put $x = 1, y = 1$ in (1), we get

$$(1)^2 + (1)^2 + 4(1) + 6(1) + k = 0$$

$$1 + 1 + 4 + 6 + k = 0$$

$$12 + k = 0$$

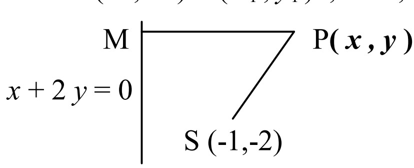
$$\boldsymbol{k = -12}$$

(1)⇒ Concentric circle is $x^2 + y^2 + 4x + 6y - \mathbf{12} = 0$

10. Find the equation of the parabola with its focus at (–1, –2) and $x + 2y = 0$ as its directrix

Answer Given focus = (–1, –2), Directrix is $x + 2y = 0$

Here $(-1, -2) = (x_1, y_1)$, $l = 1$, m = 2 , n = 0



For parabola $e = 1$

Formula: $(x - x_1)^2 + (y - y_1)^2 = e^2\left(\pm\frac{lx + my + n}{\sqrt{l^2 + m^2}}\right)^2$

$$(x+1)^2 + (y+2)^2 = 1^2\left(\pm\frac{x + 2y}{\sqrt{1^2 + (2)^2}}\right)^2$$

$$(x^2 + 1^2 + 2(x)(1)) + (y^2 + 2^2 + 2(y)(2)) = \left(\frac{x + 2y}{\sqrt{5}}\right)^2$$

$$x^2 + 1 + 2x + y^2 + 4 + 4y = \frac{(x + 2y)^2}{(\sqrt{5})^2}$$

$$x^2 + y^2 + 2x + 4y + 5 = \frac{(x + 2y)^2}{5}$$

$$5(x^2 + y^2 + 2x + 4y + 5) = (x^2 + (2y)^2 + 2(x)(2y)$$

$$5x^2 + 5y^2 + 10x + 20y + 25 - x^2 - 4y^2 - 4xy = 0$$

$$4x^2 - 4xy + y^2 + 10x + 20y + 25 = 0$$

which is the required equation of parabola

11. Find the modulus and direction cosines of the vector $2\vec{\imath} + 3\vec{\jmath} + 4\vec{k}$

Answer

Let $\vec{r} = 2\vec{\imath} + 3\vec{\jmath} + 4\vec{k} \Rightarrow x\vec{\imath} + y\vec{\jmath} + z\vec{k}$

$$|\vec{r}| = \sqrt{x^2 + y^2 + z^2}$$

$$= \sqrt{2^2 + 3^2 + 4^2}$$

$$= \sqrt{4 + 9 + 16}$$

$$Modulus\ |\vec{r}| = \sqrt{29}$$

The direction cosines is $\left(\frac{x}{|\vec{r}|}, \frac{y}{|\vec{r}|}, \frac{z}{|\vec{r}|}\right)$

$\therefore$ The direction cosines is $\left(\frac{2}{\sqrt{29}}, \frac{3}{\sqrt{29}}, \frac{4}{\sqrt{29}}\right)$

12. Find the value of 'm' if the vectors $2\vec{i} + m\vec{j} - 3\vec{k}$ and $3\vec{i} + \vec{j} + 4\vec{k}$ are perpendicular

Answer

Let $\vec{a} = 2\vec{i} + m\vec{j} - 3\vec{k}$ *and* $\vec{b} = 3\vec{i} + \vec{j} + 4\vec{k}$

$\therefore$ Condition for perpendicular $\vec{a} \bullet \vec{b} = 0$

$$(2\vec{i} + m\vec{j} - 3\vec{k}) \bullet (3\vec{i} + \vec{j} + 4\vec{k}) = 0$$

$$(2)(3) + (m)(1) + (-3)(4) = 0$$

$$6 + m - 12 = 0$$

$$m - 6 = 0$$

$$m = 6$$

13. Prove that the vectors $2\vec{i} - \vec{j} + 2\vec{k}$, $5\vec{i} + 2\vec{j} + 3\vec{k}$ *and* $4\vec{i} - 2\vec{j} + 4\vec{k}$ *are coplanar*

Answer

Let $\vec{a} = 2\vec{i} - \vec{j} + 2\vec{k}$, $\vec{b} = 5\vec{i} + 2\vec{j} + 3\vec{k}$,

$\vec{c} = 4\vec{i} - 2\vec{j} + 4\vec{k}$

Condition for coplanar $[\vec{a}, \vec{b}, \vec{c}] = 0$

$$[\vec{a}, \vec{b}, \vec{c}] = \begin{vmatrix} 2 & -1 & 2 \\ 5 & 2 & 3 \\ 4 & -2 & 4 \end{vmatrix}$$

$$= 2\begin{vmatrix} 2 & 3 \\ -2 & 4 \end{vmatrix} + 1\begin{vmatrix} 5 & 3 \\ 4 & 4 \end{vmatrix} + 2\begin{vmatrix} 5 & 2 \\ 4 & -2 \end{vmatrix}$$

$$= 2(8 + 6) + 1(20 - 12) + 2(-10 - 8)$$

$$= 2(14) + 1(8) + 2(-18)$$

$$= 28 + 8 - 36$$

$$[\vec{a}, \vec{b}, \vec{c}] = 0$$

$\therefore$ The given vectors are coplanar

14.Evaluate: $\int cos^3 x\, dx$

Answer

Formula: $cos^3 A = \frac{cos3A+3cosA}{4}$

$Put\ A = x \qquad cos^3 x = \frac{cos3x + 3cosx}{4}$

$$\int cos^3 x\, dx = \int \frac{cos3x+3cosx}{4}\, dx$$

$$= \frac{1}{4}\int (cos3x + 3cosx)\, dx$$

$$= \frac{1}{4}\left[\frac{sin3x}{3} + 3sinx\right] + c$$

15. Evaluate: $\int \frac{dx}{\sqrt{25-x^2}}$

Answer

$$\int \frac{dx}{\sqrt{25-x^2}} = \int \frac{dx}{\sqrt{5^2-x^2}}$$

$$= sin^{-1}\left(\frac{x}{5}\right) + c$$

16. Evaluate: $\int x\, e^{6x} dx$

Answer

Let $u = x$

$\frac{du}{dx} = 1$

$du = dx$

$\int dv = \int e^{6x} dx$

$v = \frac{e^{6x}}{6}$

Formula: $\int \boldsymbol{u\, dv} = \boldsymbol{uv} - \int \boldsymbol{v\, du}$

$$\int x\, e^{6x}\, dx = \frac{x\, e^{6x}}{6} - \int \frac{e^{6x}}{6}\, dx$$

$$= \frac{x\, e^{6x}}{6} - \frac{1}{6}\int e^{6x}\, dx$$

$$= \frac{x\, e^{6x}}{6} - \frac{1}{6}\left(\frac{e^{6x}}{6}\right) + c$$

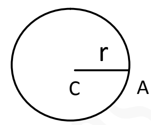
$$= \frac{x\, e^{6x}}{6} - \frac{e^{6x}}{36} + c$$

PART - C

17.(a). Find the equation of the circle passing through the Point A (1, 2) and having its centre at C (4, 6)

Answer Given A = (1, 2) C = (4, 6)

Radius $AC = \sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2}$



Here $(x_1, y_1) = (1, 2)$ and $(x_2, y_2) = (4, 6)$

$$AC = \sqrt{(4 - 1)^2 + (6 - 2)^2}$$

$$r = AC = \sqrt{(3)^2 + (4)^2}$$

$$= \sqrt{9 + 16}$$

$$= \sqrt{25} = \mathbf{5\ units}$$

Equation of circle is $(x - h)^2 + (y - k)^2 = r^2$

Put r = 5 and (h , k) = (4, 6)

$$(x - 4)^2 + (y - 6)^2 = (5)^2$$

$$(x - 4)(x - 4) + (y - 6)(y - 6) = 25$$

$$(x^2 - \mathbf{4x} - \mathbf{4x} + 16) + (y^2 - \mathbf{6y} - \mathbf{6y} + 36) = 25$$

$$x^2 - \mathbf{8x} + 16 + y^2 - \mathbf{12y} + 36 - 25 = 0$$

$$x^2 + y^2 - 8x - 12y + \mathbf{27} = 0$$

17.(b). Prove that the circles $x^2 + y^2 - 8x + 6y - 23 = 0$ and $x^2 + y^2 - 2x - 5y + 16 = 0$ cuts orthogonally

Answer

Given $\mathbf{x^2 + y^2 - 8x + 6y - 23 = 0}$ ——— (1)

$x^2 + y^2 + 2g_1x + 2f_1y + c = 0$ ——— (2)

On comparing (1) & (2), we get

$2g_1 = \mathbf{-8}$	$2f_1 = \mathbf{6}$	$c_1 = \mathbf{-23}$
$g_1 = \frac{-8}{2} = -4$	$f_1 = \frac{6}{2} = 3$	

Given $\mathbf{x^2 + y^2 - 2x - 5y + 16 = 0}$ ——— (3)

$x^2 + y^2 + 2g_2x + 2f_2y + c = 0$ ——— (4)

On Comparing (3) & (4), we get

$2g_2 = \mathbf{-2}$	$2f_2 = \mathbf{-5}$	$c_2 = \mathbf{16}$
$g_2 = \frac{-2}{2} = -1$	$f_2 = \frac{-5}{2} = -2.5$	

Condition $\mathbf{2g_1g_2 + 2f_1f_2 = c_1 + c_2}$

$2(-4)(-1) + 2(3)(-2.5) = -23 + 16$

$8 - 15 = -7$

$-7 = -7$ (satisfied)

∴ The given circles cuts orthogonally.

17.(c). Find 'k' such that the equation $2x^2 + 3xy - 2y^2 - 5x + 5y + k = 0$ represents a pair of straight lines

Answer

Given $2x^2 + 3xy - 2y^2 - 5x + 5y + k = 0$

General equation is $ax^2 + 2hxy + by^2 + 2gx + 2fy + c = 0$

a = 2	$b = -2$	$c = k$	$2h = 3$; $2g = -5$; $2f = 5$
2a = 4	$\mathbf{2b = -4}$	$\mathbf{2c = 2k}$	

Condition for pair of straight line is

$$\begin{vmatrix} 2a & 2h & 2g \\ 2h & 2b & 2f \\ 2g & 2f & 2c \end{vmatrix} = 0$$

$$\begin{vmatrix} 4 & 3 & -5 \\ 3 & -4 & 5 \\ -5 & 5 & 2k \end{vmatrix} = 0$$

$$4\begin{vmatrix} -4 & 5 \\ 5 & 2k \end{vmatrix} - 3\begin{vmatrix} 3 & 5 \\ -5 & 2k \end{vmatrix} - 5\begin{vmatrix} 3 & -4 \\ -5 & 5 \end{vmatrix} = 0$$

$$4(-8k-25) - 3(6k+25) - 5(15-20) = 0$$

$$-32k - 100 - 18k - 75 - 75 + 100 = 0$$

$$-50k - 150 = 0$$

$$-50k = 150$$

$$k = \frac{150}{-50}$$

$$k = -3$$

18.(a). Show that the points whose position vectors are

$2\vec{i} + 4\vec{j} + 3\vec{k}$, $4\vec{i} + \vec{j} - 4\vec{k}$ and $6\vec{i} + 5\vec{j} - \vec{k}$

form a right angled triangle

Answer Let $\overrightarrow{OA} = 2\vec{i} + 4\vec{j} + 3\vec{k}$, $\overrightarrow{OB} = 4\vec{i} + \vec{j} - 4\vec{k}$

$\overrightarrow{OC} = 6\vec{i} + 5\vec{j} - \vec{k}$

$\overrightarrow{AB} = \overrightarrow{OB} - \overrightarrow{OA}$

$\overrightarrow{AB} = (4\vec{i} + \vec{j} - 4\vec{k}) - (2\vec{i} + 4\vec{j} + 3\vec{k})$

$\overrightarrow{AB} = 4\vec{i} + \vec{j} - 4\vec{k} - 2\vec{i} - 4\vec{j} - 3\vec{k}$

$\overrightarrow{AB} = 2\vec{i} - 3\vec{j} - 7\vec{k}$

$|\overrightarrow{AB}| = \sqrt{(2)^2 + (-3)^2 + (-7)^2}$

$= \sqrt{4 + 9 + 49} = \sqrt{62}$

$\overrightarrow{BC} = \overrightarrow{OC} - \overrightarrow{OB}$

$\overrightarrow{BC} = (6\vec{i} + 5\vec{j} - \vec{k}) - (4\vec{i} + \vec{j} - 4\vec{k})$

$\overrightarrow{BC} = 6\vec{i} + 5\vec{j} - \vec{k} - 4\vec{i} - \vec{j} + 4\vec{k}$

$\overrightarrow{BC} = 2\vec{i} + 4\vec{j} + 3\vec{k}$

$|\overrightarrow{BC}| = \sqrt{(2)^2 + (4)^2 + (3)^2}$

$= \sqrt{4 + 16 + 9} = \sqrt{29}$

$\overrightarrow{CA} = \overrightarrow{OA} - \overrightarrow{OC}$

$\overrightarrow{CA} = (2\vec{i} + 4\vec{j} + 3\vec{k}) - (6\vec{i} + 5\vec{j} - \vec{k})$

$\overrightarrow{CA} = 2\vec{i} + 4\vec{j} + 3\vec{k} - 6\vec{i} - 5\vec{j} + \vec{k}$

$\overrightarrow{CA} = -4\vec{i} - \vec{j} + 4\vec{k}$

$|\overrightarrow{CA}| = \sqrt{(-4)^2 + (-1)^2 + (4)^2}$

$= \sqrt{16 + 1 + 16} = \sqrt{33}$

$$\Leftarrow \ |\overrightarrow{AB}|^2 = |\overrightarrow{BC}|^2 + |\overrightarrow{CA}|^2$$

$$\sqrt{62}^2 = \sqrt{29}^2 + \sqrt{33}^2$$

$$62 = 29 + 33$$

$$62 = 62$$

$\therefore$ It forms a right angled triangle

18.(b). Prove that the vectors $\vec{\imath} + 2\vec{\jmath} + \vec{k}$, $\vec{\imath} + \vec{\jmath} - 3\vec{k}$ and $7\vec{\imath} - 4\vec{\jmath} + \vec{k}$ are mutually perpendicular

Answer

Let $\vec{a} = \vec{\imath} + 2\vec{\jmath} + \vec{k}$, $\vec{b} = \vec{\imath} + \vec{\jmath} - 3\vec{k}$, $\vec{c} = 7\vec{\imath} - 4\vec{\jmath} + \vec{k}$	
$\vec{a} \bullet \vec{b} = (\vec{\imath} + 2\vec{\jmath} + \vec{k}) \bullet (\vec{\imath} + \vec{\jmath} - 3\vec{k})$ $= (1)(1) + (2)(1) + (1)(-3)$ $= 1 + 2 - 3 = \mathbf{0}$	$\therefore \vec{a} \perp \vec{b}$
$\vec{b} \bullet \vec{c} = (\vec{\imath} + \vec{\jmath} - 3\vec{k}) \bullet (7\vec{\imath} - 4\vec{\jmath} + \vec{k})$ $= (1)(7) + (1)(-4) + (-3)(1)$ $= 7 - 4 - 3 = \mathbf{0}$	$\therefore \vec{b} \perp \vec{c}$
$\vec{c} \bullet \vec{a} = (7\vec{\imath} - 4\vec{\jmath} + \vec{k}) \bullet (\vec{\imath} + 2\vec{\jmath} + \vec{k})$ $= (7)(1) + (-4)(2) + (1)(1)$ $= 7 - 8 + 1 = \mathbf{0}$	$\therefore \vec{c} \perp \vec{a}$
$\therefore$ $\vec{a}, \vec{b}$ ***and*** $\vec{c}$ **are mutually perpendicular**	

18.(c) A particle acted on by forces $3\vec{\imath} + 2\vec{\jmath} - 3\vec{k}$ and $\vec{\imath} + 7\vec{\jmath} + 7\vec{k}$ is displaced from the point $\vec{\imath} + 2\vec{\jmath} + 3\vec{k}$ to the point $3\vec{\imath} - 5\vec{\jmath} + 4\vec{k}$. Find the total work done by the forces

Answer Formula: Work done W = $\vec{F} \bullet \vec{d}$

$\vec{F} = \vec{f}_1 + \vec{f}_2$

$= (3\vec{i} + 2\vec{j} - 3\vec{k}) + (\vec{i} + 7\vec{j} + 7\vec{k})$

$= 4\vec{i} + 9\vec{j} + 4\vec{k}$

$\vec{d}$ = *To the point – From the point*

$= (3\vec{i} - 5\vec{j} + 4\vec{k}) - (\vec{i} + 2\vec{j} + 3\vec{k})$

$= 3\vec{i} - 5\vec{j} + 4\vec{k} - \vec{i} - 2\vec{j} - 3\vec{k}$

$= 2\vec{i} - 7\vec{j} + \vec{k}$

Work done W = $\vec{F} \bullet \vec{d}$

$= (4\vec{i} + 9\vec{j} + 4\vec{k}) \bullet (2\vec{i} - 7\vec{j} + \vec{k})$

$= (4)(2) + (9)(-7) + (4)(1)$

$= 8 - 63 + 4$

$= -51$

Work done W = 51 units

19.(a). Find the area of triangle whose vertices are having position vectors $3\vec{i} + 2\vec{j} - \vec{k}$, $2\vec{i} - 3\vec{j} + \vec{k}$ and $5\vec{i} + \vec{j} + 3\vec{k}$

Answer Let $\overrightarrow{OA} = 3\vec{i} + 2\vec{j} - \vec{k}$, $\overrightarrow{OB} = 2\vec{i} - 3\vec{j} + \vec{k}$

$\overrightarrow{OC} = 5\vec{i} + \vec{j} + 3\vec{k}$

Formula: Area of a triangle $= \frac{1}{2}\left|\overrightarrow{AB} \times \overrightarrow{AC}\right|$

$\overrightarrow{AB} = \overrightarrow{OB} - \overrightarrow{OA}$

$\overrightarrow{AB} = (2\vec{i} - 3\vec{j} + \vec{k}) - (3\vec{i} + 2\vec{j} - \vec{k})$

$\overrightarrow{AB} = 2\vec{i} - 3\vec{j} + \vec{k} - 3\vec{i} - 2\vec{j} + \vec{k}$

$\overrightarrow{AB} = -\vec{i} - 5\vec{j} + 2\vec{k}$

$\overrightarrow{AC} = \overrightarrow{OC} - \overrightarrow{OA}$

$\overrightarrow{AC} = (5\vec{\imath} + \vec{\jmath} + 3\vec{k}\ ) - (3\vec{\imath} + 2\vec{\jmath} - \vec{k}\ )$

$\overrightarrow{AC} = 5\vec{\imath} + \vec{\jmath} + 3\vec{k} - 3\vec{\imath} - 2\vec{\jmath} + \vec{k}$

$\overrightarrow{AC} = 2\vec{\imath} - \vec{\jmath} + 4\vec{k}$

$$\overrightarrow{AB} \times \overrightarrow{AC} = \begin{vmatrix} \vec{\imath} & \vec{\jmath} & \vec{k} \\ -1 & -5 & 2 \\ 2 & -1 & 4 \end{vmatrix}$$

$$= \vec{\imath}\begin{vmatrix} -5 & 2 \\ -1 & 4 \end{vmatrix} - \vec{\jmath}\begin{vmatrix} -1 & 2 \\ 2 & 4 \end{vmatrix} + \vec{k}\begin{vmatrix} -1 & -5 \\ 2 & -1 \end{vmatrix}$$

$$= \vec{\imath}(-20 + 2) - \vec{\jmath}(-4 - 4) + \vec{k}(1 + 10)$$

$$= \vec{\imath}(-18) - \vec{\jmath}(-8) + \vec{k}(11)$$

$$\overrightarrow{AB} \times \overrightarrow{AC} = -18\vec{\imath} + 8\vec{\jmath} + 11\vec{k}$$

$$|\overrightarrow{AB} \times \overrightarrow{AC}| = \sqrt{x^2 + y^2 + z^2} \quad \{x = -18, y = 8, z = 11\}$$

$$= \sqrt{(-18)^2 + (8)^2 + (11)^2}$$

$$= \sqrt{324 + 64 + 121}$$

$$= \sqrt{509}$$

∴**Area of a triangle** $= \frac{1}{2}|\overrightarrow{AB} \times \overrightarrow{AC}|$

$= \frac{1}{2}\sqrt{509}$ square units

19.(b). Find the moment about the point (4 , 3 , 2) of the force represented by $5\vec{\imath} - 4\vec{\jmath} + 2\vec{k}$ acting through the point (2 , 1 , − 3)

Answer Formula: Moment of the force = $\vec{r} \times \vec{F}$

Given Acting point: = (2, 1 , − 3)

About point: = (4 ,3,2)

$\vec{F}$ (force)	$\vec{F} = 5\vec{i} - 4\vec{j} + 2\vec{k}$
$\vec{r}$ = *Acting point – About point* $= (2, 1, -3) - (4, 3, 2)$ $= (2-4,\ 1-3,\ -3-2)$ $\vec{r} = (-2, -2, -5)$	

$$\vec{r} \times \vec{F} = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ -2 & -2 & -5 \\ 5 & -4 & 2 \end{vmatrix}$$

$$= \vec{i}\begin{vmatrix} -2 & -5 \\ -4 & 2 \end{vmatrix} - \vec{j}\begin{vmatrix} -2 & -5 \\ 5 & 2 \end{vmatrix} + \vec{k}\begin{vmatrix} -2 & -2 \\ 5 & -4 \end{vmatrix}$$

$$= \vec{i}(-4-20) - \vec{j}(-4+25) + \vec{k}(8+10)$$

$$= \vec{i}(-24) - \vec{j}(21) + \vec{k}(18)$$

$$\vec{r} \times \vec{F} = -24\vec{i} - 21\vec{j} + 18\vec{k}$$

19.(c). If $\vec{a} = 2\vec{i} - \vec{j} + 2\vec{k}$, $\vec{b} = \vec{i} + \vec{j} + \vec{k}$, $\vec{c} = \vec{i} + 2\vec{j} + 3\vec{k}$ and $\vec{d} = \vec{i} - \vec{j} - \vec{k}$ find $(\vec{a} \times \vec{b}) \bullet (\vec{c} \times \vec{d})$

Answer Given $\vec{a} = 2\vec{i} - \vec{j} + 2\vec{k}$, $\vec{b} = \vec{i} + \vec{j} + \vec{k}$

$$\vec{a} \times \vec{b} = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ 2 & -1 & 2 \\ 1 & 1 & 1 \end{vmatrix}$$

$$= \vec{i}\begin{vmatrix} -1 & 2 \\ 1 & 1 \end{vmatrix} - \vec{j}\begin{vmatrix} 2 & 2 \\ 1 & 1 \end{vmatrix} + \vec{k}\begin{vmatrix} 2 & -1 \\ 1 & 1 \end{vmatrix}$$

$$= \vec{i}(-1-2) - \vec{j}(2-2) + \vec{k}(2+1)$$

$$= \vec{i}(-3) - \vec{j}(0) + \vec{k}(3)$$

$$\vec{a} \times \vec{b} = -3\vec{i} - 0\vec{j} + 3\vec{k}$$

Given $\vec{c} = \vec{i} + 2\vec{j} + 3\vec{k}$, $\vec{d} = \vec{i} - \vec{j} - \vec{k}$

$$\vec{c} \times \vec{d} = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ 1 & 2 & 3 \\ 1 & -1 & -1 \end{vmatrix}$$

$$= \vec{i}\begin{vmatrix} 2 & 3 \\ -1 & -1 \end{vmatrix} - \vec{j}\begin{vmatrix} 1 & 3 \\ 1 & -1 \end{vmatrix} + \vec{k}\begin{vmatrix} 1 & 2 \\ 1 & -1 \end{vmatrix}$$

$$= \vec{i}(-2+3) - \vec{j}(-1-3) + \vec{k}(-1-2)$$

$$= \vec{i}(1) - \vec{j}(-4) + \vec{k}(-3)$$

$$\vec{c} \times \vec{d} = \vec{i} + 4\vec{j} - 3\vec{k}$$

$$(\vec{a} \times \vec{b}) \bullet (\vec{c} \times \vec{d})$$

$$= (-3\vec{i} - 0\vec{j} + 3\vec{k}) \bullet (\vec{i} + 4\vec{j} - 3\vec{k})$$

$$= (-3)(1) + (0)(4) + (3)(-3)$$

$$= -3 + 0 - 9$$

$$= -3 - 9 = -12$$

20.(a).Evaluate (i) $\int (x+1)(2x-3)\,dx$

(ii) $\int \cos 5x \cos 2x\,dx$

Answer **(i)** $\int (x+1)(2x-3)\,dx$

$$= \int (2x^2 - 3x + 2x - 3)\,dx$$

$$= \int (2x^2 - x - 3)\,dx$$

$$= \frac{2x^3}{3} - \frac{x^2}{2} - 3x + c$$

Answer **(ii)** $\int \cos 5x \cos 2x\,dx$

Formula: 2 cosA cos B = cos (A + B) + cos (A –B)

$$\cos A \cos B = \frac{1}{2}\{\cos(A+B) + \cos(A-B)\}$$

$$\int \cos 5x \cos 2x \, dx = \frac{1}{2}\int \cos(5x + 2x) + \cos(5x - 2x)\, dx$$

$$= \frac{1}{2}\int \cos 7x + \cos 3x \; dx$$

$$= \frac{1}{2}\left[\frac{\sin 7x}{7} + \frac{\sin 3x}{3}\right] + c$$

20.(b).Evaluate (i) $\int (x^2 + x + 1)^5 (2x+1)dx$ (ii) $\int \frac{\sec^2 x}{1+\tan x}\, dx$

Answer	**(i)** $\int (x^2+x+1)^5 (2x+1)\, dx$ ----- (1)
Put $t = x^2 + x + 1$ $\frac{dt}{dx} = 2x + 1$ $dt = (2x+1)dx$	$\int (x^2+x+1)^5 (2x+1)\, dx$ $= \int t^5 \, dt$ $= \frac{t^6}{6} + c \quad = \frac{(x^2+x+1)^6}{6} + c$

Answer	**(ii)** $\int \frac{\sec^2 x}{1+\tan x}\, dx$ ----- (1)
Put $t = 1 + \tan x$ $\frac{dt}{dx} = \sec^2 x$ $dt = \sec^2 x \, dx$	$\int \frac{\sec^2 x}{1+\tan x}\, dx = \int \frac{1}{t}\, dt$ $= \log(t) + c$ $= \log(1+\tan x) + c$

20.(c). Evaluate (i) $\int \frac{dx}{x^2-36}$ (ii) $\int \frac{dx}{\sqrt{49-(x+5)^2}}$

Answer (i) $\int \frac{dx}{x^2-36} = \int \frac{dx}{x^2-6^2}$

$$= \frac{1}{2(6)} log\left(\frac{x-6}{x+6}\right) + c$$

$$= \frac{1}{12} log\left(\frac{x-6}{x+6}\right) + c$$

Answer (ii) $\int \frac{dx}{\sqrt{49-(x+5)^2}} = \int \frac{dx}{\sqrt{(7)^2-(x+5)^2}}$ ----- (1)

Put $t = x + 5$ $\frac{dt}{dx} = 1$ $dt = 1\,dx$	$\int \frac{dx}{\sqrt{(7)^2-(x+5)^2}} = \int \frac{dt}{\sqrt{(7)^2-(t)^2}}$ $= sin^{-1}\left(\frac{t}{7}\right) + c$ $= sin^{-1}\left(\frac{x+5}{7}\right) + c$

21.(a). Evaluate (i) $\int x \cos 3x\, dx$ (ii) $\int x^2 \log x\, dx$

Answer (i) $\int x \cos 3x\, dx$

Let $u = x$ $\frac{du}{dx} = 1$ $du = dx$	$\int dv = \int \cos 3x\, dx$ $v = \frac{\sin 3x}{3}$

Formula: $\int u\, dv = uv - \int v\, du$

$$\int x \cos 3x\, dx = \frac{x \sin 3x}{3} - \int \frac{\sin 3x}{3} dx$$

$$= \frac{x \sin 3x}{3} - \frac{1}{3} \int \sin 3x\, dx$$

$$= \frac{x \sin 3x}{3} - \frac{1}{3}\left(\frac{-\cos 3x}{3}\right) + c$$

$$= \frac{x \sin 3x}{3} + \frac{\cos 3x}{9} + c$$

Answer **(ii) $\int x^2 \log x \, dx$**

Let $u = \log x$

$$\frac{du}{dx} = \frac{1}{x}$$

$$du = \frac{dx}{x}$$

$$\int dv = \int x^2 \, dx$$

$$v = \frac{x^3}{3}$$

Formula: $\int \boldsymbol{u \, dv} = \boldsymbol{uv} - \int \boldsymbol{v \, du}$

$$\int x^2 \log x \, dx = \log x \left(\frac{x^3}{3}\right) - \int \frac{x^3}{3} \frac{dx}{x}$$

$$= \frac{x^3}{3} \log x - \frac{1}{3} \int x^2 \, dx$$

$$= \frac{x^3}{3} \log x - \frac{1}{3} \left(\frac{x^3}{3}\right) + C$$

$$= \frac{x^3}{3} \log x - \frac{x^3}{9} + C$$

21.(b). Evaluate: $(i) \int x^2 \sin 3x \, dx$ (ii) $\int x^2 e^{5x} \, dx$

Answer **$(i) \int x^2 \sin 3x \, dx$**

Let $u = x^2$

$$u' = 2x$$

$$u'' = 2$$

$$\int dv = \int \sin 3x \, dx$$

$$v = \frac{-\cos 3x}{3}$$

$$v_1 = \frac{-\sin 3x}{9}$$

$$v_2 = \frac{\cos 3x}{27}$$

Formula: $\int \boldsymbol{u \, dv} = \boldsymbol{uv} - \boldsymbol{u' \, v_1} + \boldsymbol{u'' v_2} - \cdots$

$$\int x^2 \sin 3x \, dx = x^2 \left(\frac{-\cos 3x}{3}\right) - 2x \left(\frac{-\sin 3x}{9}\right) + \frac{2 \cos 3x}{27} + c$$

$$= \frac{-x^2 \cos 3x}{3} + \frac{2x \sin 3x}{9} + \frac{2 \cos 3x}{27} + c$$

(ii) $\int x^2 e^{5x}\, dx$

Answer

Let $u = x^2$ $u' = 2x$ $u'' = 2$	$\int dv = \int e^{5x} dx$ $v = \frac{e^{5x}}{5}$ $v_1 = \frac{e^{5x}}{25}$ $v_2 = \frac{e^{5x}}{125}$

Formula: $\int u\, dv = uv - u' v_1 + u'' v_2 - \cdots$

$$\int x^2 e^{5x} dx = x^2\left(\frac{e^{5x}}{5}\right) - 2x\left(\frac{e^{5x}}{25}\right) + 2\left(\frac{e^{5x}}{125}\right) + c$$

$$= \frac{x^2 e^{5x}}{5} - \frac{2xe^{5x}}{25} + \frac{2e^{5x}}{125} + c$$

21.(c). Evaluate: $\int_0^{\frac{\pi}{2}} (2+\sin x)^3 \cos x \; dx$

Answer $\int_0^{\frac{\pi}{2}} (2+\sin x)^3 \cos x \; dx$ ----- (1)

Put $t = 2 + \sin x$ $\frac{dt}{dx} = \cos x$ $dt = \cos x\, dx$ **Limits:** when $x = 0, t = 2 + \sin 0$ $t = 2 + 0 = 2$ when $x = \frac{\pi}{2}, t = 2 + \sin\frac{\pi}{2}$ $t = 2+1 = 3$	(1) $\Rightarrow \int_2^3 t^3\, dt$ $= \left[\frac{t^4}{4}\right]_2^3$ $= \left[\frac{3^4 - 2^4}{4}\right]$ $= \frac{81-16}{4} = \frac{65}{4}$