OCTOBER 2016

Time – Three hours

(*Maximum Marks*: 75)

[N.B:- (1) Answer any FIVE questions in each of PART-A& PART-B and any two divisions of each questions in PART-C

(2) Each questions carries 2(two)marks in PART-A ,3(three)marks in PART-B and 5(five)marks for each division in PART-C.]

PART – A

1. Find the equation of the circle whose centre is (– 5,7) and radius 3units

2. Prove that circles $x^2 + y^2 – 4x + 6y + 4 = 0$ and $x^2 + y^2 + 2x + 4y + 4 = 0$ cut orthogonally

3. If the position vectors of A and B are $3\vec{i} + 2\vec{j} - \vec{k}$ and $\vec{i} - \vec{j} + 3\vec{k}$, find $|\overrightarrow{AB}|$

4. If $|\vec{a}| = 2$, $|\vec{b}| = 7$, $|\vec{a} \times \vec{b}| = 7$, find the angle between $\vec{a}$ and $\vec{b}$

5. Find the value of $[\,\vec{i} + \vec{j}, \vec{j} + \vec{k}, \vec{k} + \vec{i}\,]$

6. Evaluate: $\int tan^2 x\, dx$

7. Evaluate: $\int \frac{dx}{4x^2+9}$

8. Evaluate: $\int_1^2 \left(x + x^2\right) dx$

PART – B

9. Find the equation of the circle passing through the point

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(5, 4) and concentric with the circle $x^2 + y^2 - 8x + 12y + 15 = 0$

10. Find the equation of the parabola with focus (1, –1) and directrix is $x - y = 0$
11. Find the projection of the vector $3\vec{i} + 4\vec{j} - 5\vec{k}$ on the vector $\vec{i} + 2\vec{j} + 2\vec{k}$
12. Find the area of the parallelogram whose adjacent sides are

    $\vec{i} + \vec{j} - 2\vec{k}$ *and* $2\vec{i} - \vec{j} - \vec{k}$
13. Evaluate: $\int (x^2 + x + 1)(x^2 - x + 1)\, dx$
14. Evaluate: $\int \frac{(\tan^{-1}x)^3}{1+x^2}\, dx$
15. Evaluate: $\int x\, sec^2 x\, dx$
16. Evaluate: $\int_0^{\frac{\pi}{2}} \frac{\cos x}{1+\sin x} dx$

PART- C

17.(a) Prove that (7,–5) lies on the circle $x^2 + y^2 - 6x + 4y - 12 = 0$

Also find the other end of the diameter through (7, –5).

(b) Prove that the circles $x^2 + y^2 - 10x - 24y + 120 = 0$ and $x^2 + y^2 = 400$ touch each other

(c) If the equation $2x^2 + 3xy - 2y^2 - 5x + 5y + C = 0$

represents a pair of straight lines, find the value of 'C'

18.(a) Prove that the points with position vectors $3\vec{i} - \vec{j} + 6\vec{k}$,

$5\vec{i} - 2\vec{j} + 7\vec{k}$ and $6\vec{i} - 5\vec{j} + 2\vec{k}$ form a right angled triangle.

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(b) Prove that the vectors $\vec{i} - \vec{j} + 2\vec{k}$, $4\vec{j} + 2\vec{k}$ and

$-10\vec{i} - 2\vec{j} + 4\vec{k}$ are mutually perpendicular

(c) If two forces $3\vec{i} + 5\vec{j} - 2\vec{k}$ and $2\vec{i} + 3\vec{j} - 5\vec{k}$ displace a particle from the point (1 , 2 , –1) to the point (5 , – 3 , 4) , find the work done by the forces

19.(a) Find the unit vector perpendicular to each of the vectors $2\vec{i} - \vec{j} + 2\vec{k}$ *and* $10\vec{i} - 2\vec{j} + 7\vec{k}$ .Also,find the sine of the angle between the two vectors

(b) Prove that the points given by the position vectors (1 , 3 , 1) , (1 , 1 , – 1) , (– 1 , 1 , 1) and (2 , 2, – 1) are coplanar

(c) If $\vec{a} = 2\vec{i} + 3\vec{j} + \vec{k}$, $\vec{b} = \vec{i} - 2\vec{j} + 3\vec{k}$ *and* $\vec{c} = 3\vec{i} - \vec{j} + 5\vec{k}$, find $(\vec{a} \times \vec{b}) \times \vec{c}$

20. (a) Evaluate: (i) $\int \frac{dx}{1+sinx}$ (ii) $\int cos5x \ cos2x \ dx$

(b) Evaluate: (i) $\int \frac{6x^2-1}{2x^3-x+5} dx$ (ii) $\int \frac{sin\sqrt{x}}{\sqrt{x}} \ dx$

(c) Evaluate: (i) $\int \frac{dx}{(5x+2)^2-9}$ (ii) $\int \frac{dx}{\sqrt{81-4x^2}}$

21.(a) Evaluate: (i) $\int x^3 \log x \, dx$ (ii) $\int x \ \sin 5x \, dx$

(b) Evaluate: (i) $\int x^2 \ \cos 3x \ dx$ (ii) $\int x^2 \, e^{-2x} \, dx$

(c) Evaluate: (i) $\int_0^1 (2x+3)^4 dx$ (ii) $\int_0^{\frac{\pi}{2}} \cos^2 x \, dx$

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ANSWERS

PART - A

1. Find the equation of the circle whose centre is (– 5 , 7) and radius 3 units

Answer Formula: $(x - \boldsymbol{h})^2 + (y - \boldsymbol{k})^2 = \mathbf{r}^2$

Here $(h, k) = (-5, 7)$ and $r = 3$

$$\therefore (x + 5)^2 + (y - 7)^2 = 3^2$$

$$(x + 5)(x + 5) + (y - 7)(y - 7) = 9$$

$$(x^2 + 5x + 5x + 25) + (y^2 - 7y - 7y + 49) = 9$$

$$x^2 + 10x + 25 + y^2 - 14y + 49 - 9 = 0$$

$$x^2 + y^2 + 10x - 14y + 65 = 0$$

2. Prove that circles $x^2 + y^2 - 4x + 6y + 4 = 0$ and $x^2 + y^2 + 2x + 4y + 4 = 0$ cut orthogonally

Answer Given $x^2 + y^2 - 4x + 6y + 4 = 0$ ——— (1)

$x^2 + y^2 + 2g_1x + 2f_1y + c_1 = 0$ ——— (2)

On comparing (1) & (2), we get

$2g_1 = -4$	$2f_1 = 6$	
$g_1 = \frac{-4}{2} = -2$	$f_1 = \frac{6}{2} = 3$	$c_1 = 4$

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Given $x^2 + y^2 + 2x + 4y + 4 = 0$ ——— (3)

$x^2 + y^2 + 2g_2x + 2f_2y + c_2 = 0$ ——— (4)

On Comparing (3) & (4), we get

$2g_2 = 2$	$2f_2 = 4$	
$g_2 = \frac{2}{2} = 1$	$f_2 = \frac{4}{2} = 2$	$c_2 = 4$

Formula: $\mathbf{2g_1g_2 + 2f_1f_2 = c_1 + c_2}$

$2(-2)(1) + 2(3)(2) = 4 + 4$

$-4 + 12 = 8$

$8 = 8$

$\therefore$ Circles cut each other orthogonally

3. If the position vectors of A and B are $3\vec{i} + 2\vec{j} - \vec{k}$ and $\vec{i} - \vec{j} + 3\vec{k}$, find $|\overrightarrow{AB}|$

Answer Let $\overrightarrow{OA} = 3\vec{i} + 2\vec{j} - \vec{k}$ and $\overrightarrow{OB} = \vec{i} - \vec{j} + 3\vec{k}$

$\overrightarrow{AB} = \overrightarrow{OB} - \overrightarrow{OA}$

$\overrightarrow{AB} = (\vec{i} - \vec{j} + 3\vec{k}) - (3\vec{i} + 2\vec{j} - \vec{k})$

$\overrightarrow{AB} = \vec{i} - \vec{j} + 3\vec{k} - 3\vec{i} - 2\vec{j} + \vec{k}$

$\overrightarrow{AB} = -2\vec{i} - 3\vec{j} + 4\vec{k}$

$|\overrightarrow{AB}| = \sqrt{(-2)^2 + (-3)^2 + (4)^2}$

$|\overrightarrow{AB}| = \sqrt{4 + 9 + 16}$

$= \sqrt{29}$ **units**

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4. If $|\vec{a}| = 2$, $|\vec{b}| = 7$, $|\vec{a} \times \vec{b}| = 7$, find the angle between $\vec{a}$ and $\vec{b}$

Answer

Given $|\vec{a}| = 2$, $|\vec{b}| = 7$, $|\vec{a} \times \vec{b}| = 7$

Formula: $\sin\theta = \dfrac{|\vec{a} \times \vec{b}|}{|\vec{a}|\,|\vec{b}|}$

$$\sin\theta = \frac{7}{(2)\,(7)}$$

$$\sin\theta = \frac{1}{2}$$

$$\Rightarrow \theta = \sin^{-1}\left(\frac{1}{2}\right)$$

$$\theta = 30^\circ$$

5. Find the value of $[\vec{i} + \vec{j}, \vec{j} + \vec{k}, \vec{k} + \vec{i}]$

Answer Given $[\vec{i} + \vec{j}, \vec{j} + \vec{k}, \vec{k} + \vec{i}]$

$$= \begin{vmatrix} 1 & 1 & 0 \\ 0 & 1 & 1 \\ 1 & 0 & 1 \end{vmatrix}$$

$$= 1\begin{vmatrix} 1 & 1 \\ 0 & 1 \end{vmatrix} - 1\begin{vmatrix} 0 & 1 \\ 1 & 1 \end{vmatrix} + 0\begin{vmatrix} 0 & 1 \\ 1 & 0 \end{vmatrix}$$

$$= 1(1 - 0) - 1(0 - 1) + 0(0 - 1)$$

$$= 1(1) - 1(-1) + 0$$

$$= 1 + 1$$

$$= 2$$

6. Evaluate: $\int \tan^2 x\, dx$

Answer Given $\int \tan^2 x\, dx = \int (\sec^2 x - 1)\, dx$

$$= \tan x - x + c$$

7. Evaluate: $\int \frac{dx}{4x^2+9}$

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Answer $\int \frac{dx}{4x^2+9} = \int \frac{dx}{(2x)^2+(3)^2}$ ----- (1)

Put $t = 2x$

$\frac{dt}{dx} = 2$

$\frac{dt}{2} = dx$

$(1) \Rightarrow \int \frac{\frac{dt}{2}}{(t)^2+(3)^2} = \frac{1}{2}\int \frac{dt}{(t)^2+(3)^2}$

$= \frac{1}{2}\left\{\frac{1}{3}tan^{-1}\left(\frac{t}{3}\right)\right\} + c$

$= \frac{1}{6}tan^{-1}\left(\frac{2x}{3}\right) + c$

8. Evaluate: $\int_1^2 (x + x^2)dx$

Answer

$$\int_1^2 (x + x^2)dx = \left[\frac{x^2}{2} + \frac{x^3}{3}\right]_1^2$$

$$= \left[\left(\frac{2^2}{2} + \frac{2^3}{3}\right) - \left(\frac{1^2}{2} + \frac{1^3}{3}\right)\right]$$

$$= \left[\left(\frac{4}{2} + \frac{8}{3}\right) - \left(\frac{1}{2} + \frac{1}{3}\right)\right]$$

$$= \left(\frac{12+16}{6}\right) - \left(\frac{3+2}{6}\right)$$

$$= \left(\frac{28}{6}\right) - \left(\frac{5}{6}\right)$$

$$= \frac{28-5}{6} = \frac{23}{6}$$

PART - B

9. Find the equation of the circle passing through the point (5, 4) and concentric with the circle $x^2 + y^2 - 8x + 12y + 15 = 0$

Answer Given $x^2 + y^2 - 8x + 12y + 15 = 0$

Concentric circle is $x^2 + y^2 - 8x + 12y + k = 0$ —— (1)

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(1) passes through (5, 4), put $x = 5$, $y = 4$ in (1), we get

$$(5)^2 + (4)^2 - 8\,(5) + 12(4) + k = 0$$

$$25 + 16 - 40 + 48 + k = 0$$

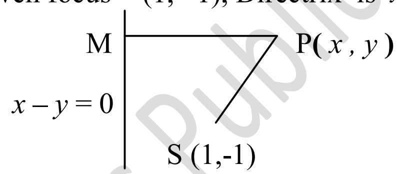
$$49 + k = 0$$

$$\boldsymbol{k = -49}$$

(1) $\Rightarrow$ Concentric circle is $x^2 + y^2 - 8x + 12y - \mathbf{49} = 0$

10. Find the equation of the parabola with focus (1, –1) and directrix is $x - y = 0$

Answer Given focus = (1, –1), Directrix is $x - y = 0$



Put $e = 1$, $(x_1, y_1) = (1, -1)$, $l = 1$, m = –1 , n = 0

Formula: $\boldsymbol{(x - x_1)^2 + (y - y_1)^2 = e^2\left(\pm\dfrac{lx + my + n}{\sqrt{l^2 + m^2}}\right)^2}$

$$(x - 1)^2 + (y + 1)^2 = 1^2\left(\pm\frac{\mathbf{x - y}}{\sqrt{\mathbf{1^2 + (-1)^2}}}\right)^2$$

$$(x^2 + 1^2 - 2(x)(1)) + (y^2 + 1^2 + 2(y)(1)) = \left(\frac{x - y}{\sqrt{2}}\right)^2$$

$$x^2 + 1 - 2x + y^2 + 1 + 2y = \frac{(x - y)^2}{(\sqrt{2})^2}$$

$$x^2 + y^2 - 2x + 2y + 2 = \frac{(x - y)^2}{2}$$

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$$2(x^2 + y^2 - 2x + 2y + 2) = (x^2 + y^2 - 2(x)(y))$$

$$2x^2 + 2y^2 - 4x + 4y + 4 - x^2 - y^2 + 2xy = 0$$

$$x^2 + 2xy + y^2 - 4x + 4y + 4 = 0$$

which is the required equation of parabola

11. Find the projection of the vector $3\vec{i} + 4\vec{j} - 5\vec{k}$ on the vector $\vec{i} + 2\vec{j} + 2\vec{k}$

Answer Let $\vec{a} = 3\vec{i} + 4\vec{j} - 5\vec{k}$, $\vec{b} = \vec{i} + 2\vec{j} + 2\vec{k}$

$$\vec{a} \bullet \vec{b} = (3\vec{i} + 4\vec{j} - 5\vec{k}) \bullet (\vec{i} + 2\vec{j} + 2\vec{k})$$

$$= (3)(1) + (4)(2) + (-5)(2)$$

$$= 3 + 8 - 10 = 1$$

$$|\vec{b}| = \sqrt{x^2 + y^2 + z^2}$$

$$= \sqrt{(1)^2 + (2)^2 + (2)^2}$$

$$= \sqrt{1 + 4 + 4}$$

$$= \sqrt{9} = 3$$

Formula: Projection of $\vec{a}$ on $\vec{b}$ = $\dfrac{\vec{a} \bullet \vec{b}}{|\vec{b}|} = \dfrac{1}{3}$ units

12. Find the area of the parallelogram whose adjacent sides are $\vec{i} + \vec{j} - 2\vec{k}$ *and* $2\vec{i} - \vec{j} - \vec{k}$

Answer Let $\vec{a} = \vec{i} + \vec{j} - 2\vec{k}$, $\vec{b} = 2\vec{i} - \vec{j} - \vec{k}$

Formula: Area $= |\vec{a} \times \vec{b}|$

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$$\vec{a} \times \vec{b} = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ 1 & 1 & -2 \\ 2 & -1 & -1 \end{vmatrix}$$

$$= \vec{i}\begin{vmatrix} 1 & -2 \\ -1 & -1 \end{vmatrix} - \vec{j}\begin{vmatrix} 1 & -2 \\ 2 & -1 \end{vmatrix} + \vec{k}\begin{vmatrix} 1 & 1 \\ 2 & -1 \end{vmatrix}$$

$$= \vec{i}(-1 - 2) - \vec{j}(-1 + 4) + \vec{k}(-1 - 2)$$

$$= \vec{i}(-3) - \vec{j}(3) + \vec{k}(-3)$$

$$\vec{a} \times \vec{b} = -3\vec{i} - 3\vec{j} - 3\vec{k}$$

$$|\vec{a} \times \vec{b}| = \sqrt{x^2 + y^2 + z^2}$$

$$= \sqrt{(-3)^2 + (-3)^2 + (-3)^2}$$

$$= \sqrt{9 + 9 + 9} = \sqrt{27} \text{ sq units}$$

13. Evaluate: $\int (x^2 + x + 1)(x^2 - x + 1)\, dx$

Answer

$$\int (x^2 + x + 1)(x^2 - x + 1)\, dx$$

$$= \int (x^4 - x^3 + x^2 + x^3 - x^2 + x + x^2 - x + 1)\, dx$$

$$= \int (x^4 + x^2 + 1)\, dx = \frac{x^5}{5} + \frac{x^3}{3} + x + c$$

14. Evaluate: $\int \frac{(\tan^{-1}x)^3}{1+x^2}\, dx$

Answer

$$\int \frac{(\tan^{-1}x)^3}{1+x^2}\, dx \text{ ----- (1)}$$

Put $t = \tan^{-1}x$ $\frac{dt}{dx} = \frac{1}{1+x^2}$ $dt = \frac{1}{1+x^2}\, dx$	$(1) \Rightarrow \int t^3\, dt$ $= \frac{t^4}{4} + c$ $= \frac{(\tan^{-1}x)^4}{4} + c$

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15. Evaluate: $\int x\, sec^2 x\, dx$

Answer

Given $\int x\, sec^2 x\, dx$

Let $u = x$

$\frac{du}{dx} = 1$

$du = dx$

$\int dv = \int sec^2 x\; dx$

$v = tanx$

Formula: $\int u\, dv = uv - \int v\, du$

$$\int x\, sec^2 x\, dx = x(tanx) - \int tanx\, dx$$

$$= xtanx - \int \frac{\sec x\, tanx}{secx} dx$$

$$= xtanx - log\,(secx) + c$$

16. Evaluate: $\int_0^{\frac{\pi}{2}} \frac{\mathbf{cosx}}{\mathbf{1+sinx}}\, \mathbf{dx}$

Answer

Formula: $\int \frac{f'(x)}{f(x)} dx = \log f(x) + c$

$$\int_0^{\frac{\pi}{2}} \frac{\cos x}{1+\sin x} dx = [\log(1+\sin x)]_0^{\frac{\pi}{2}}$$

$$= \log\left(1+\sin\frac{\pi}{2}\right) - \log(1+\sin 0)$$

$$= \log(1+1) - \log(1+0)$$

$$= \log(2\ ) - \log 1$$

$$= \log 2$$

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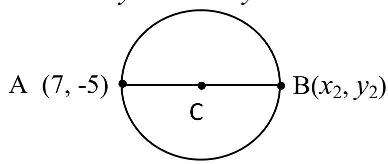
PART - C

17.(a). Prove that (7, –5) lies on the circle $x^2 + y^2 - 6x + 4y - 12 = 0$. Also find the other end of the diameter through (7, –5)

Answer Given $x^2 + y^2 - 6x + 4y - 12 = 0$ ——— (1)



Put $x = 7$, $y = -5$ in (1), we get

$$(7)^2 + (-5)^2 - 6(7) + 4(-5) - 12 = 0$$

$$49 + 25 - 42 - 20 - 12 = 0$$

$$74 - 74 = 0$$

$$0 = 0$$

$\therefore$ (7, –5) lies on the circle

To find centre : (1) $\Rightarrow x^2 + y^2 - 6x + 4y - 12 = 0$

We know that $x^2 + y^2 + 2gx + 2fy + c = 0$ ——— (2)

On comparing (1) and (2) we get

$2g = -6$	$2f = 4$	$c = -12$
$g = \frac{-6}{2} = -3$	$f = \frac{4}{2} = 2$	

$\therefore$ Centre C= $(-g, -f)$ = (3, –2)

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Let A = (7, –5) be the one end of the diameter and B (x_2, y_2) be the other end. $\therefore$ C is the midpoint of AB.

i.e., midpoint of AB = Centre C

$$\left(\frac{x_1+x_2}{2}, \frac{y_1+y_2}{2}\right) = (-g, -f)$$

$$\left(\frac{7+x}{2}, \frac{-5+y}{2}\right) = (3,-2)$$

$\frac{7+x}{2} = 3$	$\frac{-5+y}{2} = -2$
$7+x = 6$	$-5+y = -4$
$x = 6-7$	$y = -4+5$
$x = -1$	$y = 1$

$\therefore$ The other end of the diameter is (–1, 1)

17.(b). Prove that the circles $x^2 + y^2 - 10x - 24y + 120 = 0$ and $x^2 + y^2 = 400$ touch each other

Answer **Step 1:** Given $x^2 + y^2 - 10x - 24y + 120 = 0$ —— (1)

$x^2 + y^2 + 2gx + 2fy + c = 0$ ——— (2)

On comparing (1) & (2), we get

$2g = -10$	$2f = -24$	
$g = \frac{-10}{2}$	$f = \frac{-24}{2}$	
$g = -5$	$f = -12$	$c = 120$

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$\therefore$ Centre $C = (-g, -f) \Rightarrow$ $\mathbf{C_1} = (5, 12)$

$$\text{Radius } r = \sqrt{g^2 + f^2 - c}$$

$$\mathbf{r_1} = \sqrt{(-5)^2 + (-12)^2 - 120}$$

$$= \sqrt{25 + 144 - 120}$$

$$= \sqrt{49}$$

= 7 units

Step 2: Given $x^2 + y^2 - 400 = 0$ ——— (3)

$x^2 + y^2 + 2gx + 2fy + c = 0$ ——— (2)

On comparing (3) & (2), we get

$2g = 0$	$2f = 0$	
$g = 0$	$f = 0$	$c = -400$

$\therefore$ Centre $C = (-g, -f) \Rightarrow$ $\mathbf{C_2} = (0, 0)$

$$\text{Radius } r = \sqrt{g^2 + f^2 - c}$$

$$\mathbf{r_2} = \sqrt{(0)^2 + (0)^2 + 400}$$

$$= \sqrt{400}$$

= 20 units

Step 3: $d = C_1C_2 = \sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2}$

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Let $C_1 = (5, 12) = (x_1, y_1)$, $C_2 = (0, 0) = (x_2, y_2)$

$$d = \sqrt{(5-0)^2 + (12-0)^2}$$

$$= \sqrt{25+144}$$

$$= \sqrt{169} = 13 \text{ units}$$

$\therefore \quad r_2 - r_1 = 20 - 7 = 13 = d$

$$\therefore d = r_2 - r_1$$

$\therefore$ The circles touch each other internally

17.(c). If the equation $2x^2 + 3xy - 2y^2 - 5x + 5y + C = 0$ represents a pair of straight lines , find the value of ' C '

Answer Refer *April 2016 ,Question no:* 17(c) *,Page no:*11

18.(a). Show that the points with position vectors

$3\vec{\imath} - \vec{\jmath} + 6\vec{k}$, $5\vec{\imath} - 2\vec{\jmath} + 7\vec{k}$ and $6\vec{\imath} - 5\vec{\jmath} + 2\vec{k}$ form a right angled triangle

Answer Let $\overrightarrow{OA} = 3\vec{\imath} - \vec{\jmath} + 6\vec{k}$

$$\overrightarrow{OB} = 5\vec{\imath} - 2\vec{\jmath} + 7\vec{k}$$

$$\overrightarrow{OC} = 6\vec{\imath} - 5\vec{\jmath} + 2\vec{k}$$

$\overrightarrow{AB} = \overrightarrow{OB} - \overrightarrow{OA}$

$\overrightarrow{AB} = (5\vec{\imath} - 2\vec{\jmath} + 7\vec{k}) - (3\vec{\imath} - \vec{\jmath} + 6\vec{k})$

$\overrightarrow{AB} = 5\vec{\imath} - 2\vec{\jmath} + 7\vec{k} - 3\vec{\imath} + \vec{\jmath} - 6\vec{k}$

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$\overrightarrow{AB} = 2\vec{i} - \vec{j} + \vec{k}$

$|\overrightarrow{AB}| = \sqrt{(2)^2 + (-1)^2 + (1)^2} = \sqrt{4 + 1 + 1} = \sqrt{6}$

$\overrightarrow{BC} = \overrightarrow{OC} - \overrightarrow{OB}$

$\overrightarrow{BC} = (6\vec{i} - 5\vec{j} + 2\vec{k}) - (5\vec{i} - 2\vec{j} + 7\vec{k})$

$\overrightarrow{BC} = 6\vec{i} - 5\vec{j} + 2\vec{k} - 5\vec{i} + 2\vec{j} - 7\vec{k}$

$\overrightarrow{BC} = \vec{i} - 3\vec{j} - 5\vec{k}$

$|\overrightarrow{BC}| = \sqrt{(1)^2 + (-3)^2 + (-5)^2} = \sqrt{1 + 9 + 25} = \sqrt{35}$

$\overrightarrow{CA} = \overrightarrow{OA} - \overrightarrow{OC}$

$\overrightarrow{CA} = (3\vec{i} - \vec{j} + 6\vec{k}) - (6\vec{i} - 5\vec{j} + 2\vec{k})$

$\overrightarrow{CA} = 3\vec{i} - \vec{j} + 6\vec{k} - 6\vec{i} + 5\vec{j} - 2\vec{k}$

$\overrightarrow{CA} = -3\vec{i} + 4\vec{j} + 4\vec{k}$

$|\overrightarrow{CA}| = \sqrt{(-3)^2 + (4)^2 + (4)^2} = \sqrt{9 + 16 + 16} = \sqrt{41}$

$\Leftarrow |\overrightarrow{CA}|^2 = |\overrightarrow{AB}|^2 + |\overrightarrow{BC}|^2$

$\sqrt{41}^2 = \sqrt{6}^2 + \sqrt{35}^2$

$41 = 6 + 35$

$41 = 41$

$\therefore$ It forms a right angled triangle

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18.(b). Prove that the vectors $\vec{i} - \vec{j} + 2\vec{k}$, $4\vec{j} + 2\vec{k}$ and $-10\vec{i} - 2\vec{j} + 4\vec{k}$ are mutually perpendicular

Answer

Let $\vec{a} = \vec{i} - \vec{j} + 2\vec{k}$, $\vec{b} = 4\vec{j} + 2\vec{k}$, $\vec{c} = -10\vec{i} - 2\vec{j} + 4\vec{k}$	
$\vec{a} \bullet \vec{b} = (\vec{i} - \vec{j} + 2\vec{k}) \bullet (4\vec{j} + 2\vec{k})$ $= (1)(0) + (-1)(4) + (2)(2)$ $= 0 - 4 + 4 = \mathbf{0}$	$\therefore \vec{a} \perp \vec{b}$
$\vec{b} \bullet \vec{c} = (4\vec{j} + 2\vec{k}) \bullet (-10\vec{i} - 2\vec{j} + 4\vec{k})$ $= (0)(-10) + (4)(-2) + (2)(4)$ $= 0 - 8 + 8 = \mathbf{0}$	$\therefore \vec{b} \perp \vec{c}$
$\vec{c} \bullet \vec{a} = (-10\vec{i} - 2\vec{j} + 4\vec{k}) \bullet (\vec{i} - \vec{j} + 2\vec{k})$ $= (-10)(1) + (-2)(-1) + (4)(2)$ $= -10 + 2 + 8 = \mathbf{0}$	$\therefore \vec{c} \perp \vec{a}$
$\therefore$ ***$\vec{a}$, $\vec{b}$ and $\vec{c}$*** **are mutually perpendicular**	

18.(c) If two forces $3\vec{i} + 5\vec{j} - 2\vec{k}$ and $2\vec{i} + 3\vec{j} - 5\vec{k}$ displace a particle from the point (1 , 2 , –1) to the point (5 , – 3 , 4) , find the work done by the forces

Answer Formula: Work done W = $\vec{F} \bullet \vec{d}$

$\vec{F} = \vec{f}_1 + \vec{f}_2$ $= (3\vec{i} + 5\vec{j} - 2\vec{k}) + (2\vec{i} + 3\vec{j} - 5\vec{k})$ $= 5\vec{i} + 8\vec{j} - 7\vec{k}$
$\vec{d}$ = *To the point – From the point* $= (5, -3, 4) - (1, 2, -1)$ $= (5 - 1, -3 - 2, 4 + 1)$ $= (4, -5, 5) = 4\vec{i} - 5\vec{j} + 5\vec{k}$

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Work done W = $\vec{F} \bullet \vec{d}$

$= (5\vec{i} + 8\vec{j} - 7\vec{k}) \bullet (4\vec{i} - 5\vec{j} + 5\vec{k})$

$= (5)(4) + (8)(-5) + (-7)(5)$

$= 20 - 40 - 35$

$= -55$

Work done W = 55 units

19.(a). Find the unit vector perpendicular to each of the vectors $2\vec{i} - \vec{j} + 2\vec{k}$ *and* $10\vec{i} - 2\vec{j} + 7\vec{k}$. Also, find the sine of the angle between the two vectors

Answer Let $\vec{a} = 2\vec{i} - \vec{j} + 2\vec{k}$, $\vec{b} = 10\vec{i} - 2\vec{j} + 7\vec{k}$

$$\vec{a} \times \vec{b} = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ 2 & -1 & 2 \\ 10 & -2 & 7 \end{vmatrix}$$

$$= \vec{i}\begin{vmatrix} -1 & 2 \\ -2 & 7 \end{vmatrix} - \vec{j}\begin{vmatrix} 2 & 2 \\ 10 & 7 \end{vmatrix} + \vec{k}\begin{vmatrix} 2 & -1 \\ 10 & -2 \end{vmatrix}$$

$$= \vec{i}(-7 + 4) - \vec{j}(14 - 20) + \vec{k}(-4 + 10)$$

$$= \vec{i}(-3) - \vec{j}(-6) + \vec{k}(6)$$

$$\vec{a} \times \vec{b} = -3\vec{i} + 6\vec{j} + 6\vec{k}$$

$$|\vec{a} \times \vec{b}| = \sqrt{x^2 + y^2 + z^2} \quad \{x = -3, y = 6, z = 6\}$$

$$= \sqrt{(-3)^2 + (6)^2 + (6)^2}$$

$$= \sqrt{9 + 36 + 36}$$

$$|\vec{a} \times \vec{b}| = \sqrt{81}$$

$$|\vec{a}| = \sqrt{x^2 + y^2 + z^2} \quad \{x = 2, y = -1, z = 2\}$$

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$$= \sqrt{(2)^2 + (-1)^2 + (2)^2}$$

$$= \sqrt{4 + 1 + 4}$$

$$|\vec{a}| = \sqrt{9} = 3$$

$$|\vec{b}| = \sqrt{x^2 + y^2 + z^2} \; \{ x = 10 , y = -2, z = 7 \}$$

$$= \sqrt{(10)^2 + (-2)^2 + (7)^2}$$

$$= \sqrt{100 + 4 + 49}$$

$$|\vec{b}| = \sqrt{153}$$

∴The unit vector perpendicular to both $\vec{a}$ *and* $\vec{b}$ *is*

$$\hat{n} = \frac{\vec{a} \times \vec{b}}{|\vec{a} \times \vec{b}|} = \frac{-3\vec{i} + 6\vec{j} + 6\vec{k}}{\sqrt{81}}$$

∴Sine of the angle between $\vec{a}$ *and* $\vec{b}$ *is*

$$\sin\theta = \frac{|\vec{a} \times \vec{b}|}{|\vec{a}|\,|\vec{b}|}$$

$$\sin\theta = \frac{\sqrt{81}}{3\sqrt{153}}$$

$$\Rightarrow \theta = \sin^{-1}\left(\frac{\sqrt{81}}{3\sqrt{153}}\right)$$

19.(b). Prove that the points given by the position vectors (1 , 3 , 1) , (1 , 1 , – 1) , (– 1 , 1 , 1)and (2 , 2, – 1) are coplanar

Answer A,B,C,D are given points whose position vectors are

$\vec{OA} = (1 , 3 , 1) = \vec{i} + 3\vec{j} + \vec{k}$

$\vec{OB} = (1 , 1 , -1) = \vec{i} + \vec{j} - \vec{k}$

$\vec{OC} = (-1 , 1 , 1) = -\vec{i} + \vec{j} + \vec{k}$

$\vec{OD} = (2 , 2, -1) = 2\vec{i} + 2\vec{j} - \vec{k}$

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$\overrightarrow{AB} = \overrightarrow{OB} - \overrightarrow{OA}$

$\overrightarrow{AB} = (\vec{i} + \vec{j} - \vec{k}) - (\vec{i} + 3\vec{j} + \vec{k})$

$\overrightarrow{AB} = \vec{i} + \vec{j} - \vec{k} - \vec{i} - 3\vec{j} - \vec{k}$

$\overrightarrow{AB} = 0\,\vec{i} - 2\vec{j} - 2\vec{k}$

$\overrightarrow{AC} = \overrightarrow{OC} - \overrightarrow{OA}$

$\overrightarrow{AC} = (-\vec{i} + \vec{j} + \vec{k}) - (\vec{i} + 3\vec{j} + \vec{k})$

$\overrightarrow{AC} = -\vec{i} + \vec{j} + \vec{k} - \vec{i} - 3\vec{j} - \vec{k}$

$\overrightarrow{AC} = -2\,\vec{i} - 2\vec{j} + 0\,\vec{k}$

$\overrightarrow{AD} = \overrightarrow{OD} - \overrightarrow{OA}$

$\overrightarrow{AD} = (2\vec{i} + 2\vec{j} - \vec{k}) - (\vec{i} + 3\vec{j} + \vec{k})$

$\overrightarrow{AD} = 2\vec{i} + 2\vec{j} - \vec{k} - \vec{i} - 3\vec{j} - \vec{k}$

$\overrightarrow{AD} = \vec{i} - \vec{j} - 2\vec{k}$

Condition for coplanar $[\overrightarrow{AB}, \overrightarrow{AC}, \overrightarrow{AD}] = 0$

$$[\overrightarrow{AB}, \overrightarrow{AC}, \overrightarrow{AD}] = \begin{vmatrix} 0 & -2 & -2 \\ -2 & -2 & 0 \\ 1 & -1 & -2 \end{vmatrix}$$

$$= 0\begin{vmatrix} -2 & 0 \\ -1 & -2 \end{vmatrix} + 2\begin{vmatrix} -2 & 0 \\ 1 & -2 \end{vmatrix} - 2\begin{vmatrix} -2 & -2 \\ 1 & -1 \end{vmatrix}$$

$$= 0 + 2(4 + 0) - 2(2 + 2)$$

$$= 2(4) - 2(4)$$

$$= 8 - 8 = 0$$

∴ The given points A,B,C,D are coplanar.

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19.(c). If $\vec{a} = 2\vec{i} + 3\vec{j} + \vec{k}$, $\vec{b} = \vec{i} - 2\vec{j} + 3\vec{k}$ and $\vec{c} = 3\vec{i} - \vec{j} + 5\vec{k}$, find $(\vec{a} \times \vec{b}) \times \vec{c}$

Answer

Given $\vec{a} = 2\vec{i} + 3\vec{j} + \vec{k}$, $\vec{b} = \vec{i} - 2\vec{j} + 3\vec{k}$, $\vec{c} = 3\vec{i} - \vec{j} + 5\vec{k}$

$$\vec{a} \times \vec{b} = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ 2 & 3 & 1 \\ 1 & -2 & 3 \end{vmatrix}$$

$$= \vec{i}\begin{vmatrix} 3 & 1 \\ -2 & 3 \end{vmatrix} - \vec{j}\begin{vmatrix} 2 & 1 \\ 1 & 3 \end{vmatrix} + \vec{k}\begin{vmatrix} 2 & 3 \\ 1 & -2 \end{vmatrix}$$

$$= \vec{i}(9 + 2) - \vec{j}(6 - 1) + \vec{k}(-4 - 3)$$

$$= \vec{i}(11) - \vec{j}(5) + \vec{k}(-7)$$

$$\vec{a} \times \vec{b} = 11\vec{i} - 5\vec{j} - 7\vec{k}$$

$$(\vec{a} \times \vec{b}) \times \vec{c} = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ 11 & -5 & -7 \\ 3 & -1 & 5 \end{vmatrix}$$

$$= \vec{i}\begin{vmatrix} -5 & -7 \\ -1 & 5 \end{vmatrix} - \vec{j}\begin{vmatrix} 11 & -7 \\ 3 & 5 \end{vmatrix} + \vec{k}\begin{vmatrix} 11 & -5 \\ 3 & -1 \end{vmatrix}$$

$$= \vec{i}(-25 - 7) - \vec{j}(55 + 21) + \vec{k}(-11 + 15)$$

$$= \vec{i}(-32) - \vec{j}(76) + \vec{k}(4)$$

$$\vec{a} \times (\vec{b} \times \vec{c}) = -32\vec{i} - 76\vec{j} + 4\vec{k}$$

20.(a). Evaluate (i) $\int \frac{dx}{1+sinx}$ (ii) $\int cos5x\ cos2x\ dx$

Answer **(i)** $\int \frac{1}{1+sinx} dx = \int \frac{1}{(1+sinx)} \times \frac{1-sinx}{(1-sinx)} dx$

$$= \int \frac{1-sinx}{1-sin^2x} dx$$

$$= \int \frac{1-sinx}{cos^2x} dx$$

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$$= \int \left(\frac{1}{cos^2x}\right) - \left(\frac{sinx}{cos^2x}\right) dx$$

$$= \int \left(\frac{1}{cos^2x}\right) - \left(\frac{sinx}{cosx} \times \frac{1}{cosx}\right) dx$$

$$= \int (sec^2x - secx\ tanx)dx$$

$$= tanx - secx + c$$

(ii) $\int cos5x\ cos2x\ dx$

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20.(b). Evaluate (i) $\int \frac{6x^2-1}{2x^3-x+5} dx$ **(ii)** $\int \frac{sin\sqrt{x}}{\sqrt{x}} dx$

Answer **(i)** $\int \frac{6x^2-1}{2x^3-x+5} dx$ ----- (1)	
Put $t = 2x^3 - x + 5$ $\frac{dt}{dx} = 6x^2 - 1$ $dt = (6x^2 - 1)\ dx$	$(1) \Rightarrow \int \frac{1}{t}\ dt$ $= \log(t) + c$ $= \log(2x^3 - x + 5) + c$

Answer **(ii)** $\int \frac{sin\sqrt{x}}{\sqrt{x}}\ dx$ ----- (1)	
Put $t = \sqrt{x}$ $\frac{dt}{dx} = \frac{1}{2\sqrt{x}}$ $2dt = \frac{1}{\sqrt{x}}\ dx$	$(1) \Rightarrow \int sint\ (2dt) = 2\int sint\ dt$ $= 2(-\cos t) + c$ $= -2\cos(\sqrt{x}) + c$

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20.(c).Evaluate (i) $\int \frac{dx}{(5x+2)^2-9}$ **(ii)** $\int \frac{dx}{\sqrt{81-4x^2}}$

Answer (i) $\int \frac{dx}{(5x+2)^2-9} = \int \frac{dx}{(5x+2)^2-(3)^2}$ ----- (1)

Put $t = 5x + 2$

$\frac{dt}{dx} = 5$

$\frac{dt}{5} = dx$

$$(1) \Rightarrow \int \frac{\frac{dt}{5}}{(t)^2-(3)^2} = \frac{1}{5}\int \frac{dt}{(t)^2-(3)^2}$$

$$= \frac{1}{5}\left\{= \frac{1}{2(3)} log\left(\frac{t-3}{t+3}\right) + c\right\}$$

$$= \frac{1}{30} log\left(\frac{(5x+2)-3}{(5x+2)+3}\right) + c$$

$$= \frac{1}{30} log\left(\frac{5x-1}{5x+5}\right) + c$$

Answer (ii) $\int \frac{dx}{\sqrt{81-4x^2}} = \int \frac{dx}{\sqrt{(9)^2-(2x)^2}}$ ----- (1)

Put $t = 2x$

$\frac{dt}{dx} = 2$

$\frac{dt}{2} = dx$

$$(1) \Rightarrow \int \frac{\frac{dt}{2}}{\sqrt{(9)^2-(t)^2}} = \frac{1}{2}\int \frac{dt}{\sqrt{(9)^2-(t)^2}}$$

$$= \frac{1}{2} sin^{-1}\left(\frac{t}{9}\right) + c$$

$$= \frac{1}{2} sin^{-1}\left(\frac{2x}{9}\right) + c$$

21.(a). Evaluate (i) $\int x^3 \log x\, dx$ **(ii)** $\int x \sin 5x\, dx$

Answer **(i)** $\int x^3 \log x\, dx$

Let $u = \log x$

$\frac{du}{dx} = \frac{1}{x}$

$du = \frac{dx}{x}$

$dv = x^3\, dx$

$\int dv = \int x^3\, dx$

$v = \frac{x^4}{4}$

Formula: $\int u\, dv = uv - \int v\, du$

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$$\int x^3 \log x \, dx = \log x \left(\frac{x^4}{4}\right) - \int \frac{x^4}{4} \frac{dx}{x}$$

$$= \frac{x^4}{4}\log x - \frac{1}{4}\int x^3 \, dx$$

$$= \frac{x^4}{4}\log x - \frac{1}{4}\left(\frac{x^4}{4}\right) + c$$

$$= \frac{x^4}{4}\log x - \frac{x^4}{16} + c$$

Answer

(ii) $\int x \sin 5x \, dx$

Let $u = x$

$\frac{du}{dx} = 1$

$du = dx$

$\int dv = \int \sin 5x \, dx$

$v = \frac{-\cos 5x}{5}$

Formula: $\int u \, dv = uv - \int v \, du$

$$\int x \sin 5x \, dx = x\left(\frac{-\cos 5x}{5}\right) - \int \frac{-\cos 5x}{5} \, dx$$

$$= \frac{-x\cos 5x}{5} + \frac{1}{5}\int \cos 5x \, dx$$

$$= \frac{-x\cos 5x}{5} + \frac{1}{5}\left(\frac{\sin 5x}{5}\right) + c$$

$$= \frac{-x\cos 5x}{5} + \frac{\sin 5x}{25} + c$$

21.(b). Evaluate: (*i*) $\int x^2 \cos 3x \, dx$ (ii) $\int x^2 e^{-2x} \, dx$

Answer

(i) $\int x^2 \cos 3x \, dx$

Let $u = x^2$

$u' = 2x$

$u'' = 2$

$\int dv = \int \cos 3x \, dx$

$v = \frac{\sin 3x}{3}$

$v_1 = \frac{-\cos 3x}{9}$

$v_2 = \frac{-\sin 3x}{27}$

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Formula: $\int \boldsymbol{u\,dv} = \boldsymbol{uv - u'\,v_1 + u''v_2 - \cdots}$

$$\int x^2 \cos 3x\,dx = \frac{x^2 \sin 3x}{3} - 2x\left(\frac{-\cos 3x}{9}\right) + 2\left(\frac{-\sin 3x}{27}\right) + \text{c}$$

$$= \frac{x^2 \sin 3x}{3} + \frac{2x\cos 3x}{9} - \frac{2\sin 3x}{27} + c$$

Answer

(ii) $\int x^2\,\mathrm{e}^{-2x}\,dx$

Let $u = x^2$

$u' = 2x$

$u'' = 2$

$\int dv = \int e^{-2x}dx$

$v = \frac{e^{-2x}}{-2}$

$v_1 = \frac{e^{-2x}}{4}$

$v_2 = \frac{e^{-2x}}{-8}$

Formula: $\int \boldsymbol{u\,dv} = \boldsymbol{uv - u'\,v_1 + u''v_2 - \cdots}$

$$\int x^2\,\mathrm{e}^{-2x}\,dx = x^2\left(\frac{e^{-2x}}{-2}\right) - 2x\left(\frac{e^{-2x}}{4}\right) + 2\left(\frac{e^{-2x}}{-8}\right) + \text{c}$$

$$= -\frac{x^2e^{-2x}}{2} - \frac{xe^{-2x}}{2} - \frac{e^{-2x}}{4} + c$$

21.(c). Evaluate: $(i)\ \int_0^1 (2x+3)^4 dx$

(ii) $\int_0^{\frac{\pi}{2}} \cos^2 x\,dx$

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Answer (i) $\int_{0}^{1}(2x+3)^{4}dx$ ----- (1)

Put $t = 2x + 3$

$\frac{dt}{dx} = 2$

$\frac{dt}{2} = dx$

Limits:

when $x = 0, t = 2(0) + 3$

$t = 3$

when $x = 1, t = 2(1) + 3$

$t = 5$

$(1) \Rightarrow \int_{3}^{5} t^4\, dt$

$= \frac{1}{2}\int_{3}^{5} t^4\, dt$

$= \frac{1}{2}\left[\frac{t^5}{5}\right]_3^5$

$= \frac{1}{10}\left(5^5 - 3^5\right)$

$= 288.2$

Answer $\int_{0}^{\frac{\pi}{2}}\cos^2 x\,dx = \int_{0}^{\frac{\pi}{2}}\frac{1+\cos 2x}{2}dx$

$$= \frac{1}{2}\int_{0}^{\frac{\pi}{2}}(1+\cos 2x)dx$$

$$= \frac{1}{2}\left[x + \frac{\sin 2x}{2}\right]_0^{\frac{\pi}{2}}$$

$$= \frac{1}{2}\left[\left(\frac{\pi}{2} + \frac{\sin 2\frac{\pi}{2}}{2}\right) - \left(0 + \frac{\sin 2\ (0)}{2}\right)\right]$$

$$= \frac{1}{2}\left[\frac{\pi}{2} + 0 - 0\right] \quad = \frac{\pi}{4}$$