APRIL 2019

$Time - Three\ hours$

$(Maximum\ Marks: 75)$

[N.B:- (1) Answer any FIVE questions in each of PART-A& PART-B and any two divisions of each questions in PART-C

(2) Each questions carries 2(two)marks in PART-A ,3(three)marks in PART-B and 5(five)marks for each division in PART-C.]

PART – A

1. Show that the circles $x^2 + y^2 - 4x + 6y + 4 = 0$ and $x^2 + y^2 + 2x + 4y + 4 = 0$ cut orthogonally.
2. Show that the equation $x^2 - 2xy + y^2 - 16x - 12y + 22 = 0$ represents a parabola
3. Find the value of 'p' such that the vectors $2\vec{i} + \vec{j} - 5\vec{k}$ and $p\vec{i} + 3\vec{j} - 2\vec{k}$ are perpendicular.
4. If $\vec{a} = 2\vec{i} - 3\vec{j}$, $\vec{b} = \vec{i} + \vec{j} + \vec{K}$, $\vec{c} = 3\vec{i} - \vec{k}$, find $[\vec{a}\ \vec{b}\ \vec{c}]$
5. Evaluate : $\int (5x^2 - \frac{2}{x^3} + \frac{1}{x} - 3)dx$
6. Evaluate : $\int \frac{dx}{3+2x^2}$
7. Evaluate : $\int logx\ dx$
8. Evaluate: $\int_0^{\frac{\pi}{4}} \sec^2 x\, dx$

PART – B

9. Show that $2x + 3y + 9 = 0$ is a diameter of the circle $x^2 + y^2 - 6x + 10y - 1 = 0$.
10. If the position vectors of the points A and B are $\vec{i} - \vec{j} + \vec{k}$ and $3\vec{i} + 2\vec{j} + 3\vec{k}$,find$|\overrightarrow{AB}|$.Also find the direction ratio of $\overrightarrow{AB}$

11. Find 'm' if the vectors $2\vec{i}-\vec{j}+\vec{k}$, $\vec{i}+m\vec{j}-3\vec{k}$ and $3\vec{i}-\vec{j}+5\vec{k}$ are coplanar.

12. Evaluate: $\int_0^{\frac{\pi}{2}} \sin^3 x\, dx$

13. Evaluate : $\int \frac{dx}{1-sinx}$

14. Evaluate : $\int \frac{dx}{4x^2-49}$

15. Evaluate: $\int x^2 \cos 3x\, dx$

16. If $\overrightarrow{d_1} = 4\vec{i} + 2\vec{j} + 3\vec{k}$ and $\overrightarrow{d_2} = \vec{i} - \vec{j} + \vec{k}$ are diagonals of a parallelogram, find its area

PART- C

17.(a) Find the equation of the circle passing through the point $(-7, 1)$ and having its centre at $(-4, -3)$

(b) Show that the circles $x^2 + y^2 - 4x + 6y + 8 = 0$ and $x^2 + y^2 - 10x - 6y + 14 = 0$ touch each other

(c) Show that equation $2x^2 - 7xy + 3y^2 + 5x - 5y + 2 = 0$ represents a pair of straight line

18.(a) Prove that the points $2\vec{i} + 3\vec{j} + 4\vec{k}$, $3\vec{i} + 4\vec{j} + 2\vec{k}$, $4\vec{i} + 2\vec{j} + 3\vec{k}$ form an equilateral triangle.

(b) Find the projection of the vector $8\vec{i} + 4\vec{j} - 3\vec{k}$ on the vector $2\vec{i} - 3\vec{j} + 2\vec{k}$. Also find the angle between them.

(c) A particle acted on by forces $4\vec{i} + 3\vec{j} + \vec{k}$ and $2\vec{i} + 7\vec{j} - 2\vec{k}$ is displaced from the point (1,1,1) to (2,-3,5). Find the total work done

19.(a) The position vectors of the vertices of a triangle are $5\vec{i} + 2\vec{j} + 4\vec{k}$, $\vec{i} + 3\vec{j} + 2\vec{k}$ and $-\vec{i} - \vec{j} + \vec{k}$ .Find the area of the triangle.

(b) Find the moment of the force $3\vec{i} + 4\vec{j} + 5\vec{k}$ acting through a point $\vec{i} - 2\vec{j} + 3\vec{k}$ about the point $4\vec{i} - 3\vec{j} + \vec{k}$

(c) If $\vec{a} = \vec{i} + 3\vec{j} - \vec{k}$, $\vec{b} = -\vec{i} + \vec{k}$, $\vec{c} = 2\vec{i} + \vec{j}$ and $\vec{d} = 3\vec{i} - 4\vec{j} - 7\vec{k}$ find $(\vec{a} \times \vec{b}) \cdot (\vec{c} \times \vec{d})$

20.(a) Evaluate: $(i) \int \left(\sqrt{x} + \frac{1}{\sqrt{x}}\right) dx$ $(ii) \int cosx \ \ cos12x \ \ dx$

(b) Evaluate : $(i) \int \frac{cosx}{(3-5sinx)^6} dx$ $(ii) \int \frac{e^{\tan^{-1}x}}{1+x^2} dx$

(c) Evaluate : $\int \frac{dx}{3-2x-x^2}$

21.(a) Evaluate: $(i) \int x^3 \log x \, dx$ $(ii) \int x \, e^{-5x} dx$

(b) Evaluate: $(i) \int x^2 \, e^{-7x} dx$ $(ii) \int x^2 \sin 4x \, dx$

(c) Evaluate: $\int_0^{\frac{\pi}{2}} \frac{\sin^2 x}{1-\cos x} dx$

ANSWERS

PART - A

1. Show that circles $x^2 + y^2 - 4x + 6y + 4 = 0$ and $x^2 + y^2 + 2x + 4y + 4 = 0$ cut orthogonally.

***Answer Refer** October 2016, Question no:2 page no:26*

2. Show that the equation $x^2 - 2xy + y^2 - 16x - 12y + 22 = 0$ represents a parabola.

Answer Given $x^2 - 2xy + y^2 - 16x - 12y + 22 = 0$ —— (1)

We know that, $ax^2 + 2hxy + by^2 + 2gx + 2fy + c = 0$ —— (2)

On comparing (1) & (2) ,we get a = 1, b = 1, 2h = – 2 ⇒ $h = -1$

Condition for a parabola $\mathbf{h^2 - ab = 0}$

$h^2 - ab = (-1)^2 - (1)(1) = \mathbf{0}$

$\therefore$ The given conic is a parabola

3. Find the value of 'p' such that the vectors $2\vec{\imath} + \vec{\jmath} - 5\vec{k}$ and $p\vec{\imath} + 3\vec{\jmath} - 2\vec{k}$ are perpendicular.

Answer Let $\vec{a} = 2\vec{i} + \vec{j} - 5\vec{k}$, $\vec{b} = p\vec{i} + 3\vec{j} - 2\vec{k}$

$\therefore$ Condition for perpendicular $\vec{a} \bullet \vec{b} = 0$

$(2\vec{i} + \vec{j} - 5\vec{k}) \bullet (p\vec{i} + 3\vec{j} - 2\vec{k}) = 0$

$$(2)(p) + (1)(3) + (-5)(-2) = 0$$

$$2p + 3 + 10 = 0$$

$$2p + 13 = 0$$

$$p = -\frac{13}{2}$$

4. If $\vec{a} = 2\vec{i} - 3\vec{j}$, $\vec{b} = \vec{i} + \vec{j} + \vec{k}$, $\vec{c} = 3\vec{i} - \vec{k}$,

find $[\vec{a}, \vec{b}, \vec{c}]$

Answer Given $\vec{a} = 2\vec{i} - 3\vec{j}$, $\vec{b} = \vec{i} + \vec{j} + \vec{k}$, $\vec{c} = 3\vec{i} - \vec{k}$

$$[\vec{a}, \vec{b}, \vec{c}] = \begin{vmatrix} 2 & -3 & 0 \\ 1 & 1 & 1 \\ 3 & 0 & -1 \end{vmatrix}$$

$$= 2\begin{vmatrix} 1 & 1 \\ 0 & -1 \end{vmatrix} + 3\begin{vmatrix} 1 & 1 \\ 3 & -1 \end{vmatrix} + 0\begin{vmatrix} 1 & 1 \\ 3 & 0 \end{vmatrix}$$

$$= 2(-1 + 0) + 3(-1 - 3) + 0$$

$$= 2(-1) + 3(-4)$$

$$[\vec{a}, \vec{b}, \vec{c}] = -14$$

5. Evaluate: $\int \left(5x^2 - \frac{2}{x^3} + \frac{1}{x} - 3\right) dx$

Answer

$$\int \left(5x^2 - \frac{2}{x^3} + \frac{1}{x} - 3\right) dx = 5\frac{x^3}{3} - 2\frac{x^{-3+1}}{-3+1} + \log x - 3x + c$$

$$= 5\frac{x^3}{3} - 2\frac{x^{-2}}{-2} + \log x - 3x + c$$

$$= 5\frac{x^3}{3} + \frac{1}{x^2} + \log x - 3x + c$$

6. Evaluate: $\int \frac{dx}{3+2x^2}$

Answer

$$\int \frac{dx}{3+2x^2} = \int \frac{dx}{(\sqrt{3})^2+(\sqrt{2}\,x)^2} \text{----- (1)}$$

Put $t = \sqrt{2}x$ $\frac{dt}{dx} = \sqrt{2}$ $\frac{dt}{\sqrt{2}} = dx$	$(1) \Rightarrow \int \frac{\frac{dt}{\sqrt{2}}}{(\sqrt{3})^2 + (t)^2}$ $= \frac{1}{\sqrt{2}}\int \frac{dt}{(\sqrt{3})^2 + (t)^2}$ $= \frac{1}{\sqrt{2}}\left\{\frac{1}{\sqrt{3}}tan^{-1}\left(\frac{t}{\sqrt{3}}\right)\right\} + c$ $= \frac{1}{\sqrt{6}}tan^{-1}\left(\frac{\sqrt{2}x}{\sqrt{3}}\right) + c$

7. Evaluate: $\int \log x\, dx$

Answer Refer *October 2018,Question no:7 page no:122*

8. Evaluate: $\int_0^{\frac{\pi}{4}} \sec^2 x\, dx$

Answer

$$\int_0^{\frac{\pi}{4}} \sec^2 x\, dx = [tanx]_0^{\frac{\pi}{4}}$$

$$= \left(tan\frac{\pi}{4}\right) - (tan\,0)$$

$$= 1 - 0 = 1$$

PART - B

9. Show that 2x + 3y + 9 = 0 is a diameter of the circle

$x^2 + y^2 - 6x + 10y - 1 = 0$.

Answer

Given $x^2 + y^2 - 6x + 10y - 1 = 0$ ——— (1)

We know that, $x^2 + y^2 + \mathbf{2g}x + \mathbf{2f}y + \boldsymbol{c} = 0$ ——— (2)

$2g = -6$	$2f = 10$
$g = \frac{-6}{2} = -3$	$f = \frac{10}{2} = 5$

$\therefore$ Centre C = $(-g, -f) = (3, -5)$

Given equation is $2x + 3y + 9 = 0$ put $x = 3$, $y = -5$, we get

$$2(3) + 3(-5) + 9 = 0$$

$$6 - 15 + 9 = 0$$

$$0 = 0$$

$\therefore 2x + 3y + 9 = 0$ is a diameter.

10. If the position vectors of the point A and B are $\vec{\imath} - \vec{\jmath} + \vec{k}$ and $3\vec{\imath} + 2\vec{\jmath} + 3\vec{k}$, find $|\overrightarrow{AB}|$. Also find the direction ratio of $\overrightarrow{AB}$

Answer

Let $\overrightarrow{OA} = \vec{\imath} - \vec{\jmath} + \vec{k}$ and $\overrightarrow{OB} = 3\vec{\imath} + 2\vec{\jmath} + 3\vec{k}$

$\overrightarrow{AB} = \overrightarrow{OB} - \overrightarrow{OA}$

$\overrightarrow{AB} = (3\vec{\imath} + 2\vec{\jmath} + 3\vec{k}) - (\vec{\imath} - \vec{\jmath} + \vec{k})$

$\overrightarrow{AB} = 3\vec{\imath} + 2\vec{\jmath} + 3\vec{k} - \vec{\imath} + \vec{\jmath} - \vec{k}$

$\overrightarrow{AB} = 2\vec{\imath} + 3\vec{\jmath} + 2\vec{k}$

$|\overrightarrow{AB}| = \sqrt{(2)^2 + (3)^2 + (2)^2}$

$|\overrightarrow{AB}| = \sqrt{4 + 9 + 4} = \sqrt{17}$ **units**

Direction ratio of $\overrightarrow{AB}$ *are* = (2 : 3 : 2)

11. Find 'if the vectors $2\vec{\imath} - \vec{\jmath} + \vec{k}$, $\vec{\imath} + m\vec{\jmath} - 3\vec{k}$ *and* $3\vec{\imath} - 4\vec{\jmath} + 5\vec{k}$ are coplanar.

Answer Let $\vec{a} = 2\vec{i} - \vec{j} + \vec{k}$, $\vec{b} = \vec{i} + m\vec{j} - 3\vec{k}$, $\vec{c} = 3\vec{i} - 4\vec{j} + 5\vec{k}$

Condition for coplanar $[\vec{\mathbf{a}}, \vec{\mathbf{b}}, \vec{\mathbf{c}}] = \mathbf{0}$

$$\begin{vmatrix} 2 & -1 & 1 \\ 1 & m & -3 \\ 3 & -4 & 5 \end{vmatrix} = 0$$

$$2\begin{vmatrix} m & -3 \\ -4 & 5 \end{vmatrix} + 1\begin{vmatrix} 1 & -3 \\ 3 & 5 \end{vmatrix} + 1\begin{vmatrix} 1 & m \\ 3 & -4 \end{vmatrix} = 0$$

$$2(5m - 12) + 1(5 + 9) + 1(-4 - 3m) = 0$$

$$10m - 24 + 5 + 9 - 4 - 3m = 0$$

$$7m - 14 = 0$$

$$7m = 14 \quad \Rightarrow \quad m = 2$$

12. Evaluate: $\int_0^{\frac{\pi}{2}} \sin^3 x\, dx$

Answer we know that $\sin^3 A = \dfrac{3\sin A - \sin 3A}{4}$

$$\int_0^{\frac{\pi}{2}} \sin^3 x\, dx = \int_0^{\frac{\pi}{2}} \frac{3\sin x - \sin 3x}{4}\, dx$$

$$= \frac{1}{4}\int_0^{\frac{\pi}{2}} (3\sin x - \sin 3x)dx$$

$$= \frac{1}{4}\left[-3\cos x + \frac{\cos 3x}{3}\right]_0^{\frac{\pi}{2}}$$

$$= \frac{1}{4}\left[\left(-3\cos\frac{\pi}{2} + \frac{\cos 3\frac{\pi}{2}}{3}\right) - \left(-3\cos 0 + \frac{\cos 3(0)}{3}\right)\right]$$

$$= \frac{1}{4}\left[\left(-3(0) + \frac{0}{3}\right) - \left(-3(1) + \frac{1}{3}\right)\right]$$

$$= \frac{1}{4}\left[3 - \frac{1}{3}\right] = \frac{1}{4}\left[\frac{9-1}{3}\right] = \frac{1}{4}\left[\frac{8}{3}\right] = \frac{2}{3}$$

13.Evaluate: $\int \frac{1}{1-sinx}dx$

Answer

$$\int \frac{1}{1-sinx}dx = \int \frac{1}{(1-sinx)} \times \frac{1+sinx}{(1+sinx)}dx$$

$$= \int \frac{1+sinx}{1-sin^2x}dx$$

$$= \int \frac{1+sinx}{cos^2x}dx$$

$$= \int \left(\frac{1}{cos^2x}\right) + \left(\frac{sinx}{cos^2x}\right)dx$$

$$= \int \left(\frac{1}{cos^2x}\right) + \left(\frac{sinx}{cosx} \times \frac{1}{cosx}\right)dx$$

$$= \int (sec^2x + secx\ tanx)dx$$

$$= tanx + secx + c$$

14. Evaluate: $\int \frac{dx}{4x^2-49}$

Answer

Solution: $\int \frac{dx}{4x^2-49} = \int \frac{dx}{(2x)^2-(7)^2}$ ----- (1)

Put $t = 2x$ $\frac{dt}{dx} = 2$ $\frac{dt}{2} = dx$	$(1) \Rightarrow \int \frac{\frac{dt}{2}}{(t)^2 - (7)^2}$ $= \frac{1}{2}\int \frac{dt}{(t)^2 - (7)^2}$ $= \frac{1}{2}\left\{\frac{1}{2(7)} log\left(\frac{t-7}{t+7}\right)\right\} + c$ $= \frac{1}{28} log\left(\frac{2x-7}{2x+7}\right) + c$

15.Evaluate: $\int x^2 \cos 3x \, dx$

Answer Refer *October 2016,Question no:21(b.i) page no:46*

16.If $\overrightarrow{d_1} = 4\vec{i} + 2\vec{j} + 3\vec{k}$ *and* $\overrightarrow{d_2} = \vec{i} - \vec{j} + \vec{k}$ are diagonals of a parallelogram, find its area.

Answer Area of parallelogram with diagonal is $\frac{1}{2}\left|\overrightarrow{d_1} \times \overrightarrow{d_2}\right|$

$$\overrightarrow{d_1} \times \overrightarrow{d_2} = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ 4 & 2 & 3 \\ 1 & -1 & 1 \end{vmatrix}$$

$$= \vec{i}\begin{vmatrix} 2 & 3 \\ -1 & 1 \end{vmatrix} - \vec{j}\begin{vmatrix} 4 & 3 \\ 1 & 1 \end{vmatrix} + \vec{k}\begin{vmatrix} 4 & 2 \\ 1 & -1 \end{vmatrix}$$

$$= \vec{i}(2+3) - \vec{j}(4-3) + \vec{k}(-4-2)$$

$$= 5\vec{i} - \vec{j} - 6\vec{k}$$

$$\left|\overrightarrow{d_1} \times \overrightarrow{d_2}\right| = \sqrt{(5)^2 + (-1)^2 + (-6)^2} = \sqrt{62}$$

Area of parallelogram is $\frac{1}{2}\left|\overrightarrow{d_1} \times \overrightarrow{d_2}\right| = \frac{1}{2}\sqrt{62}$ sq.unit

PART-C

17 (a). Find the equation of the circle passing through the point (–7 , 1) and having its centre at (– 4 , –3)

Answer Radius $r = \sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2}$

Here $(x_1, y_1) = (-7, 1)$; $(x_2, y_2) = (-4, -3)$

$$r = \sqrt{(-4+7)^2 + (-3-1)^2}$$

$$= \sqrt{(3)^2 + (-4)^2} = \sqrt{25} = 5 \text{ units}$$

Equation of circle is $(x - h)^2 + (y - k)^2 = r^2$

Here r = 5 and (h, k) = (– 4 , –3)

$$(x + \mathbf{4})^2 + (y + \mathbf{3})^2 = (\mathbf{5})^2$$

$$(x + \mathbf{4})(x + \mathbf{4}) + (y + \mathbf{3})(y + \mathbf{3}) = 25$$

$$x^2 + 4x + 4x + 16 + y^2 + 3y + 3y + 9 = 25$$

$$x^2 + y^2 + 8x + 6y + 25 - 25 = 0$$

$$x^2 + y^2 + 8x + 6y = 0$$

which is the required equation of the circle

(b) Show that the circles $x^2 + y^2 - 4x + 6y + 8 = 0$ and $x^2 + y^2 - 10x - 6y + 14 = 0$ touch each other

Answer

Step 1: Given: $x^2 + y^2 - 4x + 6y + 8 = 0$ ——— (1)

$$x^2 + y^2 + 2gx + 2fy + c = 0 \quad \text{———} \ (2)$$

On comparing (1) & (2), we get

$2g = -4$	$2f = 6$	
$g = \frac{-4}{2} = -2$	$f = \frac{6}{2} = 3$	c = 8

$\therefore$ Centre C= (–g, –f) $\Rightarrow$ $\mathbf{C_1}$ = (2, –3)

Radius r $= \sqrt{g^2 + f^2 - c}$

$$r_1 = \sqrt{(-2)^2 + (3)^2 - 8} = \sqrt{5} \ units$$

Step 2: Given: $x^2 + y^2 - 10x - 6y + 14 = 0$ ——— (3)

$$x^2 + y^2 + 2gx + 2fy + c = 0 \quad \text{———} \ (2)$$

On comparing (3) & (2), we get

$2g = -10$	$2f = -6$	
$g = \frac{-10}{2} = -5$	$f = \frac{-6}{2} = -3$	c = 14

$\therefore$ Centre C= (–g, –f) ⇒ C_2 = (5, 3)

Radius $r = \sqrt{g^2 + f^2 - c}$

$$r_2 = \sqrt{(-5)^2 + (-3)^2 - 14} = \sqrt{20} = 2\sqrt{5} units$$

Step 3: $d = c_1c_2 = \sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2}$

Here $C_1 = (2, -3) = (x_1, y_1)$, $C_2 = (5, 3) = (x_2, y_2)$

$$d = c_1c_2 = \sqrt{(5-2)^2 + (3+3)^2}$$

$$= \sqrt{9+36}$$

$$= \sqrt{45} = 3\sqrt{5} \; units$$

$$\therefore r_1 + r_2 = \sqrt{5} + 2\sqrt{5} = 3\sqrt{5} = d$$

$$\therefore d = r_1 + r_2$$

$\therefore$ Circles touch each other externally

(c) Show that the equation $2x^2 - 7xy + 3y^2 + 5x - 5y + 2 = 0$ represents a pair of straight lines.

Answer Given $2x^2 - 7xy + 3y^2 + 5x - 5y + 2 = 0$ —— (1)

We know that, $ax^2 + 2h\,xy + by^2 + 2g\,x + 2fy + c = 0$ —— (2)

On comparing (1) & (2) ,we get

a = 2	2h = – 7	b = 3	2g = 5	2f = –5	c = 2
2a = 4		2b = 6			2c = 4

The condition for pair of straight line is $\begin{vmatrix} \mathbf{2a} & \mathbf{2h} & \mathbf{2g} \\ \mathbf{2h} & \mathbf{2b} & \mathbf{2f} \\ \mathbf{2g} & \mathbf{2f} & \mathbf{2c} \end{vmatrix} = \mathbf{0}$

$$\begin{vmatrix} 4 & -7 & 5 \\ -7 & 6 & -5 \\ 5 & -5 & 4 \end{vmatrix} = 0$$

$$4\begin{vmatrix} 6 & -5 \\ -5 & 4 \end{vmatrix} + 7\begin{vmatrix} -7 & -5 \\ 5 & 4 \end{vmatrix} + 5\begin{vmatrix} -7 & 6 \\ 5 & -5 \end{vmatrix} = 0$$

$$4(24 - 25) + 7(-28 + 25) + 5(35 - 30) = 0$$

$$4(-1) + 7(-3) + 5(5) = 0$$

$$-4 - 21 + 25 = 0$$

$$0 = 0$$

$\therefore$ The given equation represents a **pair of straight line.**

18.(a) Prove that the points form $2\vec{i}+3\vec{j}+4\vec{k}$, $3\vec{i}+4\vec{j}+2\vec{k}$, $4\vec{i}+2\vec{j}+3\vec{k}$, form an equilateral triangle.

Answer Refer *October 2017, Question no:18(a) page no:86*

(b) Find the projection of the vector $8\vec{i}+4\vec{j}-3\vec{k}$ on the vector $2\vec{i}-3\vec{j}+2\vec{k}$, Also find the angle between them.

Answer

$$\text{projection of } \vec{a} \text{ on } \vec{b} = \frac{\vec{a} \bullet \vec{b}}{|\vec{b}|}$$

Let $\vec{a} = 8\vec{i} + 4\vec{j} - 3\vec{k}$, $\vec{b} = 2\vec{i} - 3\vec{j} + 2\vec{k}$

$$\vec{a} \bullet \vec{b} = (8\vec{i} + 4\vec{j} - 3\vec{k}) \bullet (2\vec{i} - 3\vec{j} + 2\vec{k})$$

$= (8)(2) + (4)(-3) + (-3)(2)$

$= 16 - 12 - 6$

$\vec{a} \bullet \vec{b} = -2$

$|\vec{b}| = \sqrt{x^2 + y^2 + z^2}$

$= \sqrt{(2)^2 + (-3)^2 + (2)^2} = \sqrt{4 + 9 + 4} = \sqrt{17}$

$$\therefore \textbf{Projection of } \vec{a} \textbf{ on } \vec{b} = \frac{\vec{a} \bullet \vec{b}}{|\vec{b}|} = \frac{-2}{\sqrt{17}} \textbf{ units}$$

Angle between $\vec{a}$ and $\vec{b}$ is $\cos\theta = \frac{\vec{a} \bullet \vec{b}}{|\vec{a}||\vec{b}|}$

$|\vec{a}| = \sqrt{x^2 + y^2 + z^2}$

$= \sqrt{(8)^2 + (4)^2 + (-3)^2} = \sqrt{64 + 16 + 9} = \sqrt{89}$

$$\cos\theta = \frac{\vec{a} \bullet \vec{b}}{|\vec{a}||\vec{b}|}$$

$$\cos\theta = \frac{-2}{\sqrt{89}\sqrt{17}}$$

$$\Rightarrow \theta = \cos^{-1}\left(\frac{-2}{\sqrt{89}\sqrt{17}}\right)$$

(c) A particle acted on by the forces $4\vec{i} + 3\vec{j} + \vec{k}$ and $2\vec{i} + 7\vec{j} - 2\vec{k}$ is displaced from the point (1,1,1) to $(2, -3, 5)$ .Find the total work done

Answer Let Forces $\vec{f_1} = 4\vec{i} + 3\vec{j} + \vec{k}$, $\vec{f_2} = 2\vec{i} + 7\vec{j} - 2\vec{k}$

$\vec{F} = \vec{f_1} + \vec{f_2}$

$\vec{F} = (4\vec{i} + 3\vec{j} + \vec{k}) + (2\vec{i} + 7\vec{j} - 2\vec{k})$

$\vec{F} = 6\vec{i} + 10\vec{j} - \vec{k}$

From $= (1, 1, 1)$, **To** $= (2, -3, 5)$

$\vec{d}$ = **To – From**

$\vec{d} = (2, -3, 5) - (1,1,1)$

$\vec{d} = (2-1, -3-1, 5-1)$

$\vec{d} = (1, -4, 4)$

$\vec{d} = \vec{i} - 4\vec{j} + 4\vec{k}$

Work done W = $\vec{F} \bullet \vec{d}$

$= (6\vec{i} + 10\vec{j} - \vec{k}) \bullet (\vec{i} - 4\vec{j} + 4\vec{k})$

$= (6)(1) + (10)(-4) + (-1)(4)$

$= 6 - 40 - 4 = \mathbf{-38}$

Work done W = 38 units

19.(a) The position vectors of the vertices of a triangle are $5\vec{i} + 2\vec{j} + 4\vec{k}$, $\vec{i} + 3\vec{j} + 2\vec{k}$ and $-\vec{i} - \vec{j} + \vec{k}$. Find the area of the triangle.

Answer

Let $\overrightarrow{OA} = 5\vec{i} + 2\vec{j} + 4\vec{k}$, $\overrightarrow{OB} = \vec{i} + 3\vec{j} + 2\vec{k}$, $\overrightarrow{OC} = -\vec{i} - \vec{j} + \vec{k}$

Area of a triangle with position vectors: $\frac{1}{2}\left|\overrightarrow{AB} \times \overrightarrow{AC}\right|$

$\overrightarrow{AB} = \overrightarrow{OB} - \overrightarrow{OA}$

$\overrightarrow{AB} = (\vec{i} + 3\vec{j} + 2\vec{k}) - (5\vec{i} + 2\vec{j} + 4\vec{k})$

$\overrightarrow{AB} = \vec{i} + 3\vec{j} + 2\vec{k} - 5\vec{i} - 2\vec{j} - 4\vec{k}$

$\overrightarrow{AB} = -4\vec{i} + \vec{j} - 2\vec{k}$

$\overrightarrow{AC} = \overrightarrow{OC} - \overrightarrow{OA}$

$\overrightarrow{AC} = (-\vec{i} - \vec{j} + \vec{k}) - (5\vec{i} + 2\vec{j} + 4\vec{k})$

$\overrightarrow{AC} = -\vec{i} - \vec{j} + \vec{k} - 5\vec{i} - 2\vec{j} - 4\vec{k}$

$\overrightarrow{AC} = -6\vec{i} - 3\vec{j} - 3\vec{k}$

$$\overrightarrow{AB} \times \overrightarrow{AC} = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ -4 & 1 & -2 \\ -6 & -3 & -3 \end{vmatrix}$$

$$= \vec{i}\begin{vmatrix} 1 & -2 \\ -3 & -3 \end{vmatrix} - \vec{j}\begin{vmatrix} -4 & -2 \\ -6 & -3 \end{vmatrix} + \vec{k}\begin{vmatrix} -4 & 1 \\ -6 & -3 \end{vmatrix}$$

$$= \vec{i}(-3 - 6) - \vec{j}(12 - 12) + \vec{k}(12 + 6)$$

$$\overrightarrow{AB} \times \overrightarrow{AC} = -9\vec{i} + 0\vec{j} + 18\vec{k}$$

$$\left|\overrightarrow{AB} \times \overrightarrow{AC}\right| = \sqrt{x^2 + y^2 + z^2} \quad \{x = -9, y = 0, z = 18\}$$

$$= \sqrt{(-9)^2 + (0)^2 + (18)^2}$$

$$= \sqrt{81 + 0 + 324} = \sqrt{405}$$

Area of a triangle $= \frac{1}{2}\left|\overrightarrow{AB} \times \overrightarrow{AC}\right| = \frac{1}{2}\sqrt{405}$ square units

(b) Find the moment of the force $3\vec{i} + 4\vec{j} + 5\vec{k}$ acting through a point $\vec{i} - 2\vec{j} + 3\vec{k}$ about the point $4\vec{i} - 3\vec{j} + \vec{k}$.

Answer

Given $\vec{F} = 3\vec{i} + 4\vec{j} + 5\vec{k}$

Acting point: $\overrightarrow{OA} = \vec{i} - 2\vec{j} + 3\vec{k}$

About point: $\overrightarrow{OP} = 4\vec{i} - 3\vec{j} + \vec{k}$

Moment $= \vec{r} \times \vec{F}$

$$\vec{r} = \overrightarrow{PA} = \overrightarrow{OA} - \overrightarrow{OP}$$

$$= (\vec{i} - 2\vec{j} + 3\vec{k}) - (4\vec{i} - 3\vec{j} + \vec{k})$$

$$= \vec{i} - 2\vec{j} + 3\vec{k} - 4\vec{i} + 3\vec{j} - \vec{k}$$

$$\vec{r} = -3\vec{i} + \vec{j} + 2\vec{k}$$

$$\vec{r} \times \vec{F} = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ -3 & 1 & 2 \\ 3 & 4 & 5 \end{vmatrix}$$

$$= \vec{i}\begin{vmatrix} 1 & 2 \\ 4 & 5 \end{vmatrix} - \vec{j}\begin{vmatrix} -3 & 2 \\ 3 & 5 \end{vmatrix} + \vec{k}\begin{vmatrix} -3 & 1 \\ 3 & 4 \end{vmatrix}$$

$$= \vec{i}(5-8) - \vec{j}(-15-6) + \vec{k}(-12-3)$$

$$= \vec{i}(-3) - \vec{j}(-21) + \vec{k}(-15)$$

$$\vec{r} \times \vec{F} = -3\vec{i} + 21\vec{j} - 15\vec{k}$$

(c) If $\vec{a} = \vec{i} + 3\vec{j} - \vec{k}$, $\vec{b} = -\vec{i} + \vec{k}$, $\vec{c} = 2\vec{i} + \vec{j}$ and $\vec{d} = 3\vec{i} - 4\vec{j} - 7\vec{k}$, find $(\vec{a} \times \vec{b}) \cdot (\vec{c} \times \vec{d})$

Answer Given $\vec{a} = \vec{i} + 3\vec{j} - \vec{k}$, $\vec{b} = -\vec{i} + \vec{k}$

$$\vec{a} \times \vec{b} = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ 1 & 3 & -1 \\ -1 & 0 & 1 \end{vmatrix}$$

$$= \vec{i}\begin{vmatrix} 3 & -1 \\ 0 & 1 \end{vmatrix} - \vec{j}\begin{vmatrix} 1 & -1 \\ -1 & 1 \end{vmatrix} + \vec{k}\begin{vmatrix} 1 & 3 \\ -1 & 0 \end{vmatrix}$$

$$= \vec{i}(3-0) - \vec{j}(1-1) + \vec{k}(0+3)$$

$$= \vec{i}(3) - \vec{j}(0) + \vec{k}(3)$$

$$= 3\vec{i} + 3\vec{k}$$

Given $\vec{c} = 2\vec{i} + \vec{j}$, $\vec{d} = 3\vec{i} - 4\vec{j} - 7\vec{k}$

$$\vec{c} \times \vec{d} = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ 2 & 1 & 0 \\ 3 & -4 & -7 \end{vmatrix}$$

$$= \vec{i}\begin{vmatrix} 1 & 0 \\ -4 & -7 \end{vmatrix} - \vec{j}\begin{vmatrix} 2 & 0 \\ 3 & -7 \end{vmatrix} + \vec{k}\begin{vmatrix} 2 & 1 \\ 3 & -4 \end{vmatrix}$$

$$= \vec{i}(-7+0) - \vec{j}(-14+0) + \vec{k}(-8-3)$$

$$= \vec{i}(-7) - \vec{j}(-14) + \vec{k}(-11)$$

$$\vec{c} \times \vec{d} = -7\vec{i} + 14\vec{j} - 11\vec{k}$$

$$(\vec{a} \times \vec{b}) \cdot (\vec{c} \times \vec{d}) = (3\vec{i} + 3\vec{k}) \cdot (-7\vec{i} + 14\vec{j} - 11\vec{k})$$

$$= (3)(-7) + (0)(14) + (3)(-11)$$

$$= -21 + 0 - 33 = -54$$

20. (a) Evaluate: (i) $\int\left(\sqrt{x}+\frac{1}{\sqrt{x}}\right)dx$ (ii) $\int cosx\ cos12x\ dx$

Answer

(i) $\int\left(\sqrt{x}+\frac{1}{\sqrt{x}}\right)dx = \int\left(x^{\frac{1}{2}}+\frac{1}{\sqrt{x}}\right)dx$

$$= \frac{x^{\frac{1}{2}+1}}{\frac{1}{2}+1} + 2\sqrt{x} + c$$

$$= \frac{x^{\frac{3}{2}}}{\frac{3}{2}} + 2\sqrt{x} + c$$

$$= \frac{2x^{\frac{3}{2}}}{3} + 2\sqrt{x} + c$$

(ii) $\int cosx\ cos12x\ dx = \int cos12x\ cosx\ dx$

we know that cosA cosB $=\frac{1}{2}$ { cos(A+ B) + cos(A – B)}

$$\int cos12x\ cosx\ dx = \frac{1}{2}\int cos(12x+x) + cos(12x-x)\,dx$$

$$= \frac{1}{2}\int cos13x + cos11x\,dx$$

$$= \frac{1}{2}\left[\frac{sin13x}{13} + \frac{sin11x}{11}\right] + c$$

(b) . Evaluate (i) $\int \frac{cosx}{(3+5sinx)^6}dx$ (ii) $\int \frac{e^{\tan^{-1}x}}{1+x^2}dx$

Answer	$(i)\ \int \frac{cosx}{(3+5sinx)^6}dx$----- (1)
Put $t = 3 + 5\sin x$ $\frac{dt}{dx} = 5\cos x$ $\frac{dt}{5} = \cos x dx$	$(1) \Rightarrow \int \frac{\frac{dt}{5}}{t^6} = \frac{1}{5}\int \frac{dt}{t^6}$ $= \frac{1}{5}\left(\frac{t^{-5}}{-5}\right) + c$ $= \frac{1}{5}\left(\frac{(3+5\sin x)^{-5}}{-5}\right) + c$

(ii) $\int \frac{e^{\tan^{-1}x}}{1+x^2}dx$

Answer	$\int \frac{e^{\tan^{-1}x}}{1+x^2}dx$----- (1)
Put $t = \tan^{-1}x$ $\frac{dt}{dx} = \frac{1}{1+x^2}$ $dt = \frac{1}{1+x^2}dx$	$(1) \Rightarrow \int e^t\, dt$ $= e^t + c$ $= e^{\tan^{-1}x} + c$

(c) Evaluate: $\int \frac{dx}{3-2x-x^2}$

Answer Consider $3 - 2x - x^2 = -(x^2 + 2x - 3)$

$$= -(x^2 + 2x + 1 - 1 - 3)$$

$$= -((x + 1)^2 - 4)$$

$$= -((x + 1)^2 - 2^2)$$

$$= 2^2 - (x + 1)^2$$

$$\int \frac{dx}{3 - 2x - x^2} = \int \frac{dx}{2^2 - (x + 1)^2}$$

$$= \frac{1}{2(2)} log\left(\frac{2 + (x + 1)}{2 - (x + 1)}\right)$$

$$= \frac{1}{4} log\left(\frac{3+x}{1-x}\right) + c$$

21.(a) Evaluate:(i) $\int x^3 \log x\, dx$ **(ii)** $\int x\, e^{-5x} dx$

(i) $\int x^3 \log x\, dx$

Answer Refer *October 2016, Question no:21a (i) page no:45*

(ii) $\int x\, e^{-5x} dx$

Answer

Let $u = x$

$$\frac{du}{dx} = 1$$

$$du = dx$$

$$dv = e^{-5x}dx$$

$$\int dv = \int e^{-5x}dx$$

$$v = \frac{e^{-5x}}{-5}$$

We know that $\int \boldsymbol{udv} = \boldsymbol{uv} - \int \boldsymbol{vdu}$

$$\int xe^{-5x}dx = \frac{xe^{-5x}}{-5} - \int \frac{e^{-5x}}{-5}dx$$

$$= \frac{xe^{-5x}}{-5} + \frac{1}{5}\int e^{-5x}dx$$

$$= \frac{xe^{-5x}}{-5} + \frac{1}{5}\left(\frac{e^{-5x}}{-5}\right) + c$$

$$= \frac{xe^{-5x}}{-5} - \frac{e^{-5x}}{25} + c$$

(b) Evaluate:(i) $\int x^2 \mathrm{e}^{-7x}dx$ (ii) $\int x^2 \sin 4x\, dx$

(i) $\int x^2 \mathrm{e}^{-7x}dx$

Answer

Let $u = x^2$

$u' = 2x$

$u'' = 2$

$$dv = e^{-7x}dx$$

$$\int dv = \int e^{-7x}dx$$

$$v = \frac{e^{-7x}}{-7}$$

$$v_1 = \frac{e^{-7x}}{49}$$

$$v_2 = \frac{e^{-7x}}{-343}$$

We know that $\int \boldsymbol{u\,dv} = \boldsymbol{uv} - \boldsymbol{u'v_1} + \boldsymbol{u''v_2} - \cdots$

$$\int x^2\mathrm{e}^{-7\mathrm{x}}dx = x^2\left(\frac{e^{-7x}}{-7}\right) - 2x\left(\frac{e^{-7x}}{49}\right) + 2\left(\frac{e^{-7x}}{-343}\right) + \mathrm{c}$$

$$= -\frac{x^2e^{-3x}}{7} - \frac{2xe^{-3x}}{49} - \frac{2e^{-3x}}{343} + c$$

(ii) $\int x^2 \sin 4x\, dx$

Answer

Let $u = x^2$
$u' = 2x$
$u'' = 2$

$dv = \sin 4x \, dx$

$\int dv = \int \sin 4x \, dx$

$v = \frac{-\cos 4x}{4}$

$v_1 = \frac{-\sin 4x}{16}$

$v_2 = \frac{\cos 4x}{64}$

We know that $\int \boldsymbol{u\, dv = uv - u'v_1 + u''v_2} - \cdots$

$$\int x^2 \sin 4x \, dx = x^2\left(\frac{-\cos 4x}{4}\right) - 2x\left(\frac{-\sin 4x}{16}\right) + \frac{2\cos 4x}{64} + c$$

$$= \frac{-x^2\cos 4x}{4} + \frac{2x\sin 4x}{16} + \frac{2\cos 4x}{64} + c$$

(c) .Evaluate: $\int_0^{\frac{\pi}{2}} \frac{\sin^2 x}{1-\cos x} dx$

Answer

$$\int_0^{\frac{\pi}{2}} \frac{\sin^2 x}{1-\cos x} dx = \int_0^{\frac{\pi}{2}} \frac{\sin^2 x(1+\cos x)}{(1-\cos x)(1+\cos x)} dx$$

$$= \int_0^{\frac{\pi}{2}} \frac{\sin^2 x(1+\cos x)}{1-\cos^2 x} dx$$

$$= \int_0^{\frac{\pi}{2}} \frac{\sin^2 x\,(1+\cos x)}{\sin^2 x} dx$$

$$= \int_0^{\frac{\pi}{2}} (1+\cos x) dx$$

$$= [x+\sin x]_0^{\frac{\pi}{2}}$$

$$= \left(\frac{\pi}{2}+\sin\frac{\pi}{2}\right) - (0+\sin 0)$$

$$= \frac{\pi}{2}+1$$